Jurnal Elemen, 10(3), 501-515, September 2024
https:/doi.org/10.29408/jel.v10i3.25278

e- ISSN: 2442 - 4226

# Challenges in teaching students to plot equations: Another impact of graphing procedures

**Ulumul Umah *, Ana Rahmawati**

Mathematics Education Department, Universitas Pesantren Tinggi Darul 'Ulum, East Java, Indonesia

* Correspondence: ulumul.umah@unipdu.ac.id


## Abstract

Graphs are essential representations in mathematics, but many students only focus on procedural rules when sketching graphs of equations. This study aims to describe the errors and obstacles experienced by students in drawing graphics of equations. The research method uses a qualitative approach by collecting data through tests and interviews. The research results show that students' mistakes in drawing graphs can be caused by dependence on procedural knowledge in plotting points without conceptual understanding. Students who make mistakes do not understand points as multiplicative objects, lack covariational reasoning, need help understanding the concept of complex numbers, and show reflective thinking in completing assignments. This research implies that teachers must consider the students' difficulties and mistakes to avoid becoming didactic learning obstacles. In studying plotting point procedures, students' knowledge must include understanding concepts related to graphs, such as the meaning of points and graph concavity, graph shifts, and the relationship between the discriminant of a quadratic equation and its graph.

**Keywords:** covariational reasoning; didactical obstacles; graphs; mathematical understanding

**How to cite:** Umah, U., & Rahmawati, A. (2024). Challenges in teaching students to plot equations: Another impact of graphing procedures. *Jurnal Elemen*, *10*(3), 501-515. https://doi.org/10.29408/jel.v10i3.25278

Received: 2 February 2024 | Revised: 21 March 2024
Accepted: 29 April 2024 | Published: 14 September 2024

## Introduction

Graphs are an essential mathematics representation, but student errors in constructing and interpreting graphs still need to be corrected in learning. The ability to represent certain functions, such as formulas, tables, and graphs, and switch between these representations is part of understanding functions (Carlson, 1998). Students still have many limitations in understanding the relationship between algebraic and graphical representations, namely equations and graphs (Birgin, 2012; Knuth, 2020; Veloo et al., 2015). Drawing graphs also includes fundamental challenges in critical thinking in calculus (Susilo et al., 2021).

Drawing graphs involves both procedural and conceptual knowledge. Skemp (1976) distinguished two types of mathematical understanding: relational and instrumental. Relational understanding is defined as "knowing what to do and why," and rational mathematics learning process as "building conceptual structures," while instrumental understanding can be said as "rules without reason" (Barmby et al., 2007). These two types of understanding are also often associated with conceptual and procedural knowledge, which cannot be separated and must be a concern in designing learning strategies (Hurrell, 2021). When drawing a graph, one may only use procedural knowledge to pair input-output values from equations and conceive the graph as a geometric figure (Oehrtman et al., 2008). It may also involve knowledge of slope, covariation, and graph shift concepts in the Cartesian plane. Drawing graphs only with procedural knowledge can indicate a limited understanding of function, namely limited to the action level (Breidenbach et al., 1992). With the action view, students can input values into algebraic expressions and calculate them but will have difficulty dealing with more common situations, such as composing two functions based on graphs when the algebraic expression is not given. Procedural knowledge without meaningful learning will cause problems, while conceptual knowledge can avoid sources of learning interference (Hurrell, 2021).

Limitations of students' mathematical understanding lead to errors in solving mathematical problems. Student errors in mathematical concepts can occur due to learning obstacles from previous learning experiences. In the context of learning, the phrases "error" and "obstacle" indicate distinct things. "Error" refers to a mistake or inaccuracy committed during the learning process. It could be a misperception, misinterpretation, or misunderstanding of a notion or concept (Barbieri et al., 2019). An error is primarily formed within surface levels of knowledge, so a child's response to a task is procedural and can be corrected by the teacher providing correct alternatives (Ryan & Williams, 2007). Errors can be addressed and enhance knowledge acquisition if learners can cope with them adaptively and reflectively (Metcalfe, 2017). An "obstacle," on the other hand, is anything that prevents a learner from fully participating in the learning process.

Conru (2002) categorizes learning obstacles as ontogenetic, epistemological, and didactic. The learner's cognitive development causes the ontogenetic obstacle, while the epistemological obstacle occurs due to the interference of certain mathematical concepts with more complicated concepts. Meanwhile, didactic obstacles arise due to the nature of the teaching or teacher. Didactic obstacles are known when educators introduce representations or other teaching resources but result in inconsistency of students' ideas on more complex mathematics learning objectives, which these barriers can and should be avoided (Cortina et al., 2003). The manifestation of mathematical didactic obstacles is an error (Brousseau, 2002).

Plotting points is one of the strategies used in drawing graphs of equations and functions. It is commonly taught in high schools. In this strategy, students will look for graph intersection points on the $x$ and $y$ axes or choose a value for $x$ to be substituted into the equation (Knuth, 2020). This procedure may work well for some routine problems but can cause errors in other cases, such as $y = x^2 + 1$, namely when it cannot find the coordinates of the intersection points based on algebraic expressions. Thus, the algebraic procedure alone to determine the point of intersection of the axes is not enough without reasoning.

Teaching procedures without reasoning can be a source of didactic obstacles for students learning about graphics. Teacher expertise may hinder the implementation of meaningful practice in graphic drawing competence (Glazer, 2011). Although indications of a lack of mathematical understanding can be identified through a test, limited student understanding may not always be detected during learning at school, especially if the teacher only involves routine problems. A standalone test has limitations as a tool for assessing mathematical understanding in that students can answer the test correctly without understanding (Barmby et al., 2007). Problems of understanding mathematics that cannot be overcome during learning in high school will be carried over when students enter college.

Drawing graphics is related to understanding multiplicative objects. The multiplicative object in question is different from the multiplication operation. Frank (2017) illustrates that a multiplicative object as a mental action enables a student to imagine two quantities varying together to satisfy some invariant relationship. In the context of graphs, Thompson (2011) gives an example of a point in the Cartesian plane as a multiplicative object that unites the point's distance from the horizontal axis to the point's distance from the vertical axis.

Drawing graphs also involves covariational reasoning, namely the ability to coordinate values on two variables simultaneously (Carlson et al., 2002), which is also related to understanding multiplicative objects. Thompson & Carlson (2017) explain covariational reasoning in two ways: paying attention to reasoning variations separately from covariational reasoning or considering how to construct multiplicative objects from quantity values. To understand covariational reasoning correctly, one must maintain persistent attention on two quantities simultaneously (Whitmire, 2014), which requires the ability to construct multiplicative objects. Frank (2016) and Frank (2017) suggest supporting students in constructing coordinates as multiplicative objects and testing the development of reasoning in graphical contexts. Based on the experimental results, it was found that graphs, as a type of graphical representation, are more effective in helping to understand functions in terms of covariation (Rolfes et al., 2021).

Previous studies have described difficulties in understanding graphs (Birgin, 2012; Knuth, 2020; Susilo et al., 2021; Veloo et al., 2015) but have not provided a perspective on plotting point procedures as a didactic obstacle. While Knuth (2020) has also shown that there is an error as an effect of understanding that relies too much on algebraic procedures rather than graphs, this research is limited to linear equations. Several studies have been conducted on difficulties in reasoning about covariation but have not explained further its relationship to previous learning practices (Sandie et al., 2019; Sutini et al., 2020; Syarifuddin et al., 2020; Whitmire, 2014; Zeytun et al., 2010). Meanwhile, didactic obstacles in learning mathematics were also studied, including the topic of fractions (Cortina et al., 2003), decimal numbers (Brousseau, 2002), transformation geometry (Sunariah & Mulyana, 2020), and online learning (Murniasih et al., 2022).

Plotting points by checking the intersection of the graph with the axes is a helpful procedure in sketching a graph. This procedure is widely taught by teachers in schools. Previous studies have shown that many students rely too much on routine procedures, which leads to errors when constructing or interpreting graphs. However, prior research still has limitations in explaining the impact of the plotting point procedure without a conceptual understanding of students' ability to draw graphs in a broad context. This study uses the plotting point procedure to describe the students' didactic obstacles in drawing graphs. The results of this study are expected to be useful theoretically in explaining student errors in graphs and the obstacles that cause these errors. Practically, the research results can be used by teachers to improve learning strategies on graphs in secondary schools.

## Methods

A qualitative approach was used in this study to describe the types of student errors in drawing graphs and analyze the didactic obstacles of teaching about graphics with the plotting point procedure that causes these errors. The plotting point referred to in this study includes the method for determining coordinate points or intersection points for making graphic sketches. In this research, the researcher was the main instrument that collected and analyzed the data. The researchers were present openly to obtain in-depth information, and their status as researchers was known. Data was collected on students as pre-service teachers at the University of Pesantren Tinggi Darul Ulum. All participants' names were left anonymous to comply with research ethics. There were three students as participants in this research. They were selected based on criteria that applied value input procedures with tables or drawing pairs of points discretely but made errors in expressing them in graphs.

Data collection was carried out through tests and interviews. There were two test items. The indicators of the test items are: (1) students can represent graphs of equations using the procedural method of finding intersection points on the axes, (2) students can represent graphs of equations that do not have intersection points on the x-axis without dependence on the input-output value calculation procedure. The test is an assignment to draw a graph of a linear equation $y = x + 6$ for indicator (1) and a quadratic equation with roots of complex numbers $y = x^2 + 1$ for indicator (2). Quadratic equations are given to investigate students' ability to draw graphs when the procedure for finding graph intersection points is complicated only with algebraic calculations. Meanwhile, linear equations are given to check student perceptions of coordinate points as multiplicative objects. Instrument validation was tested through peer review between mathematics education lecturers.

Furthermore, interviews were conducted to investigate and confirm students' understanding and habits when faced with the task of drawing graphs. The qualitative data analysis began by reading the written response data and interview transcripts thoroughly, then giving meaning to each part of the data and writing codes. Each associated code was grouped into themes and described to explain the presence of errors and related learning obstacles. To ensure the validity and reliability of the findings, researchers compared data with codes repeatedly to ensure consistency of meaning and involved discussions with external auditors regarding the quality of the analysis.

## Results

This section will explain three cases of students who experience errors in drawing graphs: NN, FA, and UR. The three subjects chosen were third-year female students from the same class where all the participants were women. Female students were selected to minimize the possibility of bias because only one male student was in the class. These three cases were selected from students who made mistakes in drawing graphs and represent the variety of cases found. Another consideration is their communication skills and willingness to participate in research. NN and UR are students with relatively high abilities in their classes and have high perseverance and motivation in solving problems. However, both show different errors, which will be displayed in the next section. Meanwhile, FA has relatively moderate abilities in class and is highly motivated to complete assignments even though she sometimes experiences some difficulties.

The three subjects showed different types of mistakes. However, all three used the plotting point procedure at the intersection of the axes or specific points before starting to draw the graph. Each subject's response will be explained as follows.

**NN case**

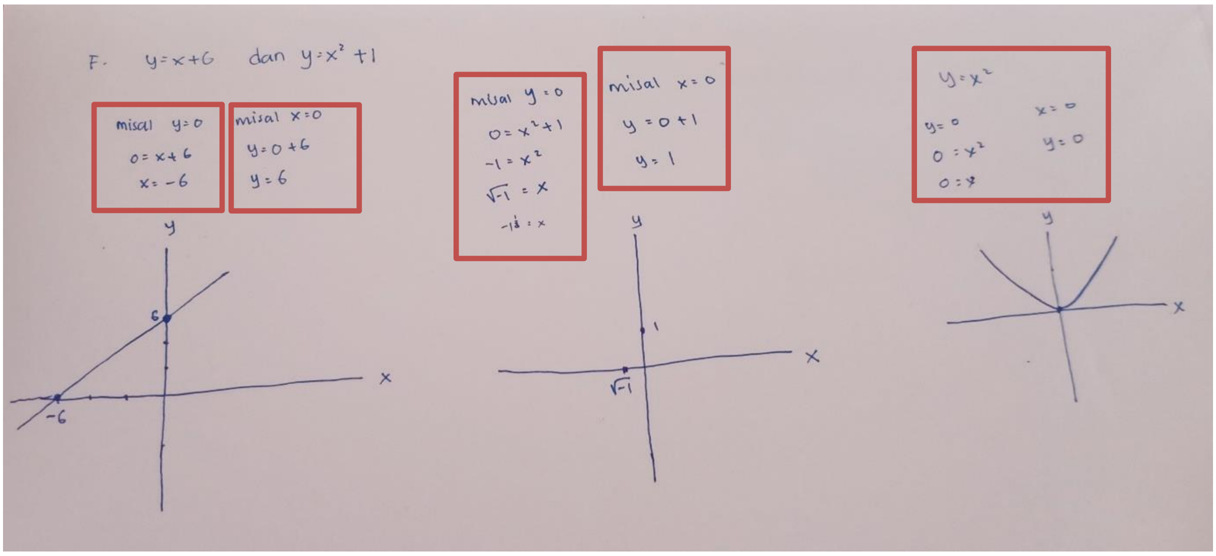


**Figure 1.** Student NN can draw a graph for $y = x + 6$ and $y = x^2$ but has difficulty doing so on a graph $y = x^2 + 1$ because it cannot represent the point of intersection of the graph with the $x$-axis.

Student NN starts her work by determining the point of intersection of the graph with the $x$-axis and $y$-axis. She can perform this procedure in drawing linear graphs. However, she stalled and encountered a problem when graphing a quadratic equation with complex roots, as shown in Figure 1. She could not graph the equation $y = x^2 + 1$ because she could not determine the intersection point of the graph with the $x$-axis. The results of the interview also revealed that she had difficulty drawing a graph of $y = x^2 + 1$ because she could not represent the complex number $\sqrt{-1}$ in the Cartesian plane. Although she did write $\sqrt{-1}$ on the $x$-axis, which was a mistake due to a lack of understanding of complex number representations; she was unsure of the answer and, therefore, did not draw a graph based on this error. However, when asked to graph $y = x^2$ she can do so. This subject does not use the idea of shifting the graph of $y = x^2$ to produce the graph of $y = x^2 + 1$. Even though she can only determine one intersection point, namely (0,0) on the graph of $y = x^2$, she can draw a graph based on previous learning experiences in drawing a graph of a quadratic equation with a minimum point at (0,0).

**FA case**

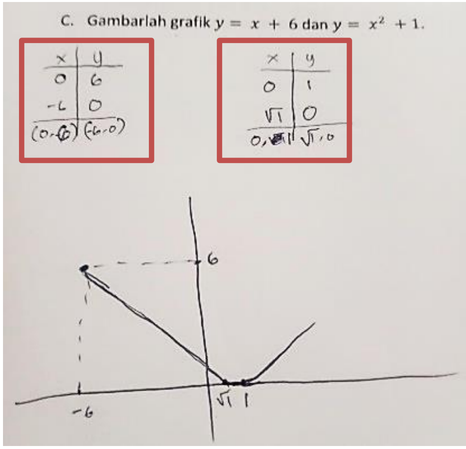


**Figure 2.** Student FA also performed the procedure for taking intersection points and made mistakes in representing coordinate points.

Student FA experienced errors in drawing graphs in linear and quadratic equations. She wrote a strategy for determining the intersection points of the axes but made a mistake in selecting the coordinates in linear and quadratic equations. She combines the graphs of the equations $y = x^2$ and $y = x^2 + 1$, as shown in Figure 2. This subject experiences an error in understanding a point in the Cartesian plane as a multiplicative object that unites the point's distance from the horizontal axis to the point's distance from the vertical axis (Figure 3). Another mistake that FA made was an algebraic error in determining the value of $x$ when $x^2 + 1 = 0$ by stating the value of $x = 1$.

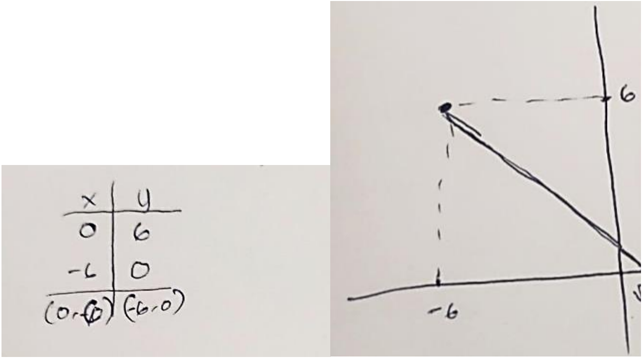

**Figure 3.** FA determines the point of intersection of the axes through input values in algebraic expressions but cannot represent it correctly in the graph.

Two students, namely NN and FA, stated that the way to draw a graphic sketch is to find the intersection points, which can be seen from the statements of the subjects in the interviews, including "to draw a graph, we need to find the intersection points with the x and y axes." In the following case, it will be shown that student errors occur not only when they determine the intersection points of the axes but also when they determine pairs of dots discretely and calculate them in algebraic equations.

**UR case**

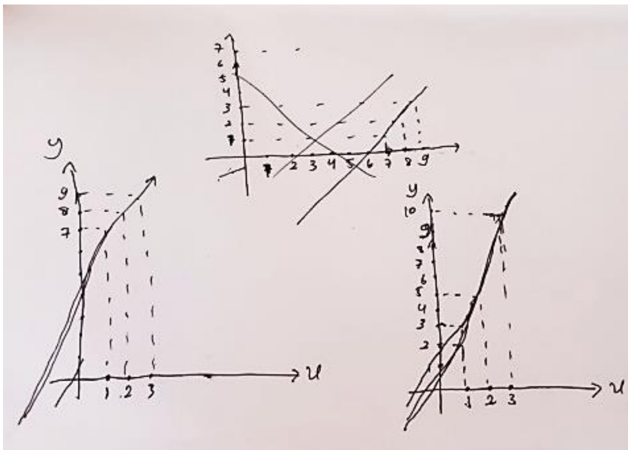

**Figure 4.** Student UR discretely performs the input procedure of several value pairs and connects the points with line segments.

Student UR describes the graph by inputting several cases of variable value pairs that satisfy the equation. Next, on the graph $y = x^2 + 1$ UR also inputs several pairs of variable values and connects these points with line segments, as shown in Figure 4. When confirmed through interviews, UR stated that the quadratic equation graph is indeed a straight line. This student does not consider the concept of continuously changing the slope of a line. The following are excerpts from an interview with UR.

Instructor : For this second graph (quadratic equation), is the graph in the form of a straight line like this?

UR : Yes, ma'am

Instructor : How do I get this?

UR : We enter the *x* values, for example, 1, 2, and 3, to make the dots so that it becomes a graph like this.

The types of errors experienced by the students illustrate the learning obstacles they experienced (Table 1). The first two students, NN and FA, encountered obstacles when they applied the plotting point procedure to determine the graph's intersection point with the axis, but without being based on the required conceptual understanding. The obstacle experienced by the third subject occurred because the subject carried out the procedure for inputting variable values without coordinating the magnitude of changes in the two variables simultaneously. Value input procedures are usually carried out by determining the table of values before sketching the graph.

**Table 1.** Types of errors and obstacles experienced by the students.

| Student | Error | Obstacle |
|---|---|---|
| NN | Error in graph a quadratic equation that has complex roots. | Unable to determine the $x$-axis intersect point when using the graph-intersects procedure. |
| FA | The graphs of linear and quadratic equations are incorrect, and the coordinates are inaccurate. | Carrying out procedures to find the graph's intersection point with the axis, but not understanding the point as a multiplicative object and lack of ability in algebraic operations. |
| UR | Assuming that the graph of the quadratic equation is a composite of straight-line segments based on the input of several value pairs | Continuous and smooth covariational reasoning limitations |

## Discussion

For the first case, student NN can draw a graph for $y = x + 6$ and $y = x^2$ but has difficulty doing so on a graph $y = x^2 + 1$. This student can fluently represent graphs of equations by determining graph intersection points. However, there is a problem when this procedure cannot be carried out on the given quadratic equation. The obstacle experienced by the student was not being able to represent the x-axis intersection point for quadratic equations whose roots are complex numbers. The difficulties with graph understanding were closely related to prerequisite skills such as basic mathematical concepts (Andalia et al., 2020). On the other hand, the findings in NN show that basic skills alone are not enough for students to represent graphs of all types of equations. A global process of perceiving the graph as a whole is required to understand graphs, including identifying patterns and intervals where values increase or decrease (Bragdon et al., 2019).

Student FA also performed the procedure for taking intersection points and made mistakes in representing coordinate points. FA determines the point of intersection of the axes through input values in algebraic expressions but cannot describe it correctly in the graph. These results indicate that FA experiences obstacles in thinking about points as multiplicative objects (Frank, 2016). The meaning of dots as multiplicative objects greatly influences students' graphing activities (Tasova et al., 2020; Tasova & Moore, 2021). Perceiving multiplicative objects is the basis for conducting covariational reasoning on graphs (Frank, 2016; Stevens et al., 2017).

The results indicate that student NN and FA relied on the procedure of determining the intersection of the axes to draw a graph. Student NA encountered obstacles, namely, does not understand the connection between the nature of square roots or the discriminant of quadratic equations and their graphical representations and does not apply the graph translation strategy $y = x^2$ that she already knows. Student responses often depend heavily on the algebraic solution method and are unaware of the simpler graphical solution method (Knuth, 2020). Meanwhile, student FA experienced further obstacles and could not represent the intersection points in a visual representation in the Cartesian plane. Difficulties translating representations can be seen based on the dimensions of fact gaps, attribute density, and confounding concepts (Bossé et al., 2011). The student's difficulties can be categorized as fact gaps, namely when they cannot determine the intersection point with the $x$-axis on the graph $y = x^2 + 1$.

Student UR discretely performs the input procedure of several value pairs and connects the points with line segments. The student has an obstacle where she has limited covariational reasoning, which has not yet reached the smooth and continuous level (Castillo-Garsow et al., 2013; Thompson & Carlson, 2017). Student UR does not understand the concept of continuously changing the slope of a line, which is also often experienced in other cases (Cho & Nagle, 2017). She lacks continuous and smooth covariational reasoning (Thompson & Carlson, 2017) by only being able to coordinate discrete values and not taking into account changes in values between the two pairs of values he chooses. Glazer (2011) states that errors in drawing graphs are often encountered, including understanding graphs as constructing discrete points. This finding aligns with other research findings, which state that university students have difficulty understanding graphs, especially in predicting values that are not given (Bragdon et al., 2019).

Students in this study have difficulty moving from algebraic symbolic representations to graphical representations. The problem of difficulty translating symbolic representations into graphics is one of the difficulties in school mathematics (Nurrahmawati et al., 2021). The problem of students' difficulties in switching between representations can be related to the curriculum and teaching of algebra in high schools, which only presents limited issues that can be solved with symbolic representation frameworks (Knuth, 2020).

An understanding of other mathematical concepts and the connections between these concepts must accompany the application of the plotting point procedure. This study found that the lack of knowledge of complex numbers and their operations and the relationship between the discriminant of quadratic equations and graphical representations prevented students from drawing proper graphs. This finding shows that weak mathematical connection abilities result in a lack of ability to draw graphs. Complete mathematical understanding can be assessed by the connections between concepts (Barmby et al., 2007). In school mathematics learning, teachers need to make students have meaningful mathematical connections (Mhlolo et al., 2012). Mathematical connections also play a role in successful problem-solving (Retnawati et al., 2020).

Deficiencies in covariational reasoning and understanding of points as multiplicative objects are also a source of student difficulties. Learning to emphasize covariational reasoning should start in middle school because students need to study topics in university mathematics (Carlson et al., 2002; Ellis et al., 2020). Covariational reasoning assessments must also be conducted at the secondary school level (Cavey et al., 2019). This reasoning is a prerequisite knowledge in calculus that students must master (Carlson et al., 2015).

The obstacles mentioned above can occur due to teaching that only focuses on procedural knowledge without being accompanied by conceptual knowledge. Conceptual understanding must be mastered from the start to prevent the occurrence of didactic obstacles from the practice of learning about graphics through plotting point procedures. Moving from conceptual to procedural understanding will provide many opportunities for students to develop knowledge, but moving from the opposite direction will risk causing errors (Hurrell, 2021). Although conceptual knowledge is essential to successful learning, it does not mean that procedural knowledge can be ignored. Both of this knowledge must be mastered simultaneously. Procedural and conceptual knowledge should proceed iteratively until complete understanding is achieved (Rittle-Johnson et al., 2001). The results of this study indicate that students' didactic obstacles occur at various levels of procedural understanding. Students who also experience difficulties in algebraic operating procedures experience more significant challenges in dealing with assigned tasks. Both procedural and conceptual knowledge can have different levels of depth (Hurrell, 2021). Other studies also show that working memory also plays a role in mathematical achievement along with procedural skills and conceptual understanding (Gilmore et al., 2017), which means it may also be related to ontogenetic obstacles (Manrique et al., 2024).

Some mistakes are sometimes not realized by students, such as in the case of UR, who was not aware of the visual characteristics of the graph of a quadratic equation, in contrast to NN, who was aware of difficulties even though she failed in overcoming these difficulties. Chin et al. (2002) found that students cannot always express their problems or doubts; they are not even aware of what they do not know or understand. Teachers need to know how to treat student errors because this plays a role in gaining meaningful knowledge (Zohar & Zohar, 2004). A lack of awareness of wrong answers can also indicate a lack of reflective thinking, a characteristic of critical thinking (Ennis, 1993). Therefore, these didactic obstacles must be overcome first so that students' thinking can move towards increasing critical thinking.

## Conclusion

Students experience learning obstacles based on previous experience in 'plotting point ' procedure lessons. They can carry out this procedure to calculate and determine pairs of values at specific points, but they do not always successfully sketch a graph correctly representing the overall equation. The consequences of this obstacle manifest in the form of graphic drawing errors made by students. Student errors in drawing graphs included not being able to correctly represent coordinate points due to the lack of understanding of multiplicative objects. Students also experience difficulties representing graphs that do not intersect the -axis due to relying too much on algebraic symbolic representation calculations. The lack of understanding of complex numbers, the connection between the discriminant of quadratic equations and graphical representations, and the lack of ability to operate algebra also hinder students from constructing graphical representations, which are considered obstacles. Another obstacle is the lack of covariational reasoning, which also causes students to represent graphs of quadratic equations with line segments with different slopes. Overall, the learning barriers experienced by students in drawing graphs of equations can result from teaching 'plotting point' procedures that need to be integrated with knowledge of the network of concepts connected to understanding the meaning of mathematical representations as a whole. The findings of this study do not mean that the plotting point procedure causes errors in drawing graphs. The research results prove that this procedure should be taught alongside conceptual knowledge. This procedural knowledge must still be taught by considering the findings of didactic obstacles in this study.

The plotting point procedure, namely determining the table of value pairs or the intersection point with the $x$-axis, must be taught by considering the difficulties and mistakes of the students mentioned above so they do not become didactic learning obstacles. Therefore, in teaching plotting point procedures, the teacher must also include knowledge of graphics-related concepts and the relationships between these concepts. It is also essential to introduce a graph shift strategy to overcome obstacles when it is difficult to determine the point of intersection with the axis.

Other obstacles and errors that differ from these findings may be discovered through further study. The context of using the point plotting procedure tested in this research is limited to linear and quadratic equations. The tests given in follow-up research can also be further developed on different types of equations.

## Acknowledgement

We sincerely thank the Directorate General of Higher Education, Ministry of Education and Culture of the Republic of Indonesia, for their generous funding of this research through the 2023 Lecturer Research Program. This support has been instrumental in the successful completion of our study. Additionally, we thank Universitas Pesantren Tinggi Darul Ulum for providing essential facilities and resources that have significantly contributed to advancing our research endeavors. The collaboration and support from these institutions have played a crucial role in the successful execution of this project, and we are truly grateful for their valuable contributions.

## Conflicts of Interest

The authors declare no conflict of interest regarding the publication of this manuscript. In addition, the authors have completed the ethical issues, including plagiarism, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies.

## Funding Statement

This work was supported by the Ministry of Education, Culture, Research, and Technology of Indonesia [grant number 0557/E5.5/AL.04/2023].

## Author Contributions

**Ulumul Umah:** Conceptualization, writing - original draft, editing, and visualization; **Ana Rahmawati:** Writing - review & editing, formal analysis, and methodology.

## References

Andalia, N., AG, B., & Zulfajri, M. (2020). The student's ability in graph understanding for mastering natural science concepts through the process skills approach. *International Journal of Instruction*, *13*(4), 145–160.

Barbieri, C. A., Miller-Cotto, D., & Booth, J. L. (2019). Lessening the load of misconceptions: Design-based principles for algebra learning. *Journal of the Learning Sciences*, *28*(3), 381–417.

Barmby, P., Harries, T., Higgins, S., & Suggate, J. (2007). How can we assess mathematical understanding. *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education*, 41–48.

Birgin, O. (2012). Investigation of eighth-grade students' understanding of the slope of the linear function. *Bolema - Mathematics Education Bulletin*, *26*(42 A), 139–162. https://doi.org/10.1590/S0103-636X2012000100008

Bossé, M. J., Adu-Gyamfi, K., & Cheetham, M. R. (2011). Assessing the difficulty of mathematical translations: Synthesizing the literature and novel findings. *International Electronic Journal of Mathematics Education*, *6*(3), 113–133. https://doi.org/10.29333/iejme/264

Bragdon, D., Pandiscio, E., & Speer, N. (2019). University students' graph interpretation and comprehension abilities. *Investigations in Mathematics Learning*, *11*(4), 275–290.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*(3), 247–285. https://doi.org/10.1007/BF02309532

Brousseau, G. (2002). Epistemological obstacles, problems, and didactical engineering. In B. Nicolas, C. Martin, S. Rosamund, & W. Virginia (Ed.), *Theory of Didactical Situations in Mathematics* (Nomor 19, hal. 79–117). https://doi.org/10.1007/0-306-47211-2_6

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. *Research in Collegiate Mathematics Education*, *1*(7), 115–162. https://doi.org/10.1090/cbmath/007/04

Carlson, M. P., Jacobs, S., Coe, E., Larsen, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. *Journal for* *research in mathematics education*, *33*(5), 352–378. https://doi.org/10.2307/4149958

Carlson, M. P., Madison, B., & West, R. D. (2015). A study of students' readiness to learn calculus. *International Journal of Research in Undergraduate Mathematics Education*, *1*(2), 209–233. https://doi.org/10.1007/s40753-015-0013-y

Castillo-Garsow, C., Johnson, H. L., & Moore, K. C. (2013). Chunky and smooth images of change. *For the Learning of Mathematics*, *33*(3), 31–37.

Cavey, L. O., Totorica, T., Libberton, J., Carney, M., Souders, K., & Lowenthal, P. (2019). A framework for analyzing secondary students' covariational reasoning. *Annual research conference of the National Council of Teachers of Mathematics*.

Chin, C., Brown, D. E., & Bruce, B. C. (2002). Student-generated questions: A meaningful aspect of learning in science. *International Journal of Science Education*, *24*(5), 521–549. https://doi.org/10.1080/09500690110095249

Cho, P., & Nagle, C. (2017). Procedural and conceptual difficulties with slope: an analysis of students' mistakes on routine tasks. *International Journal of Research in Education and Science (IJRES)*, *3*(1), 135–150. https://ijres.net/index.php/ijres/article/view/142

Conru, B. (2002). Limits. In D. Tall (Ed.), *Advanced mathematical thinking* (Vol. 11, hal. 153–165). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47203-1_10

Cortina, J. L., Visnovska, J., & Zúñiga, C. (2003). Equipartition as a didactical obstacle in fraction instruction. *Acta Didactica Universitatis Comenianae Mathematics*, *May 2016*, 1–17.

Ellis, A., Ely, R., Singleton, B., & Tasova, H. (2020). *Scaling-continuous variation : supporting students' algebraic reasoning*.

Ennis, R. H. (1993). Critical thinking assessment. *Theory Into Practice*, *32*(3), 179–186. https://doi.org/10.1080/00405849309543594

Frank, K. . (2016). Plotting points: implications of "over and up" on students' covariational reasoning. *Proceedings of the 38th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, *November*, 573–580.

Frank, K. M. (2017). Examining the development of students' covariational reasoning in the context of graphing. In *Dissertation*. Arizona State University.

Gilmore, C., Keeble, S., Richardson, S., & Cragg, L. (2017). The interaction of procedural skill, conceptual understanding and working memory in early mathematics achievement. *Journal of Numerical Cognition*, *3*(2).

Glazer, N. (2011). Challenges with graph interpretation: A review of the literature. *Studies in Science Education*, *47*(2), 183–210. https://doi.org/10.1080/03057267.2011.605307

Hurrell, D. (2021). Conceptual knowledge or procedural knowledge or conceptual knowledge and procedural knowledge: Why the conjunction is important to teachers. *Australian Journal of Teacher Education*, *46*(2), 57–71. https://doi.org/10.14221/ajte.2021v46n2.4

Knuth, E. J. (2020). Understanding connections between equations and graphs. *The Mathematics Teacher*, *93*(1), 48–53. https://doi.org/10.5951/mt.93.1.0048

Manrique, H. M., Read, D. W., & Walker, M. J. (2024). On some statistical and cerebral aspects of the limits of working memory capacity in anthropoid primates, with particular reference to pan and homo, and their significance for human evolution. *Neuroscience & Biobehavioral Reviews*, 105543.

Metcalfe, J. (2017). Learning from errors. *Annual review of psychology*, *68*, 465–489.

Mhlolo, M. K., Schafer, M., & Venkat, H. (2012). The nature and quality of the mathematical connections teachers make. *pythagoras*, *33*(1), 1–9.

Murniasih, T. R., Suwanti, V., Syaharuddin, S., Rahaju, R., & Farida, N. (2022). Prospective teachers' perceptions of didactic obstacles in the online mathematics learning. *Jurnal Elemen*, *8*(2), 619–630. https://doi.org/10.29408/jel.v8i2.5740

Nurrahmawati, Sa'dijah, C., Sudirman, & Muksar, M. (2021). Assessing students' errors in mathematical translation: From symbolic to verbal and graphic representations. *International Journal of Evaluation and Research in Education*, *10*(1), 115–125.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Ed.), *Making the Connection: Research and Teaching in Undergraduate Mathematics Education* (hal. 27–42). Mathematical Association of America.

Retnawati, H., Apino, E., & Santoso, A. (2020). High school students' difficulties in making mathematical connections when solving problems. *International Journal of Learning, Teaching and Educational Research*, *19*(8), 255–277.

Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, *93*(2), 346.

Rolfes, T., Roth, J., & Schnotz, W. (2021). Mono - and multi-representational learning of the covariational aspect of functional thinking. *Journal for STEM Education Research*, 1–27. https://doi.org/10.1007/s41979-021-00060-4

Ryan, J., & Williams, J. (2007). *Children's mathematics 4-15: Learning from errors and misconceptions: learning from errors and misconceptions*. McGraw-Hill Education (UK).

Sandie, Purwanto, Subanji, & Hidayanto, E. (2019). Student difficulties in solving covariational problems. *International Journal of Humanities and Innovation (IJHI)*, *2*(2), 42–47. https://doi.org/10.33750/ijhi.v2i2.38

Skemp, R. R. (1976). Relational understanding and instrumental understanding. *Mathematics Teaching*, *77*, 20–26. https://doi.org/10.5951/at.26.3.0009

Stevens, I. E., Paoletti, T., Moore, K. C., Liang, B., & Hardison, H. (2017). Principles for designing tasks that promote covariational reasoning. *Proceeding of the 20th Annual Conference on Research in Undergraduate Mathematics Education*, 928–936.

Sunariah, L., & Mulyana, E. (2020). The didactical and epistemological obstacles on the topic of geometry transformation. *Journal of Physics: Conference Series*, *1521*(3). https://doi.org/10.1088/1742-6596/1521/3/032089

Susilo, B. E., Darhim, D., & Prabawanto, S. (2021). Students' learning difficulties in integral calculus based on critical thinking skills. *Journal of Physics: Conference Series*, *1918*(4), 042058. https://doi.org/10.1088/1742-6596/1918/4/042058

Sutini, Aaidati, I. F., & Kusaeri. (2020). Identifying the structure of students' argumentation in covariational reasoning of constructing graphs. *Beta: Jurnal Tadris Matematika*, *13*(117), 61–80. https://doi.org/10.20414/betajtm.v13i1.374

Syarifuddin, S., Nusantara, T., Qohar, A., & Muksar, M. (2020). Students' thinking processes connecting quantities in solving covariation mathematical problems in high school students of indonesia. *Participatory Educational Research*, *7*(3), 59–78. https://doi.org/10.17275/per.20.35.7.3

Tasova, H. I., Liang, B., & Moore, K. C. (2020). *The role of lines and points in the construction of emergent shape thinking*.

Tasova, H. I., & Moore, K. C. (2021). *Framework for representing a multiplicative object in the context of graphing*. *December*, 236–245. https://doi.org/10.51272/pmena.42.2020-24

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Ed.), *New perspectives and directions for collaborative research in mathematics education. WISDOMe Monographs* (Nomor 1, hal. 33–57).

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (hal. 421–456). Compendium for research in mathematics education.

Veloo, A., Krishnasamy, H. N., & Wan Abdullah, W. S. (2015). Types of student errors in mathematical symbols, graphs and problem-solving. *Asian Social Science*, *11*(15), 324–334. https://doi.org/10.5539/ass.v11n15p324

Whitmire, B. J. (2014). *Undergraduate students' development of covariational reasoning*. Arizona State University.

Zeytun, A. S., Cetinkaya, B., & Erbas, A. K. (2010). Mathematics teachers' covariational reasoning levels and predictions about students' covariational reasoning abilities. *Educational Sciences: Theory and Practice*, *10*(3), 1601–1612.

Zohar, A., & Zohar, A. (2004). Teachers' knowledge about the treatment of students' wrong answers. *Higher Order Thinking in Science Classrooms: Students' Learning and Teachers' Professional Development*, 139–156.