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# Redesigning geometry assessments to promote advanced geometric thinking: A case study on formal deduction and rigor

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**Abstract**

Many geometry problems at the university level, particularly in analytic geometry courses, tend to prioritize procedural tasks to foster deeper geometric thinking. This study aims to analyze and redesign existing geometry problems to enhance pre-service mathematics teachers' formal deductive reasoning and rigor in accordance with the Van Hiele model. Employing a case study approach, four geometric problems were analyzed in relation to their alignment with various levels of geometric thinking. The study involved a detailed examination of pre-service mathematics teachers' responses and the structure of their problems to identify aspects that require improvement to better support higher-order thinking. The methodology included a content analysis of problem design and pre-service mathematics teachers' answers, focusing on their engagement in formal deduction and generalization. The findings indicate that the current problems insufficiently promote the development of formal deduction and rigor, as they are primarily centered on formula applications without requiring proof or generalization. Specific recommendations are provided in the form of redesigned analytic geometric problems aimed at fostering advanced geometric thinking. These redesigns are expected to help pre-service mathematics teachers tackle more complex mathematical problems by encouraging logical reasoning and argumentation.

**Keywords**: analytic geometry; geometric thinking; redesign; reasoning; van Hiele model

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## Introduction

The mathematics education curriculum, as outlined in the Indonesian Mathematics Educators Society (I-MES) and the Common Core State Standards for Mathematics (CCSSM) documents, requires pre-service mathematics teachers to have a deep mastery of geometry concepts. This includes high-level geometric thinking skills such as formal deduction, generalization, and mathematical proof. In analytic geometry courses, students are expected not only to understand procedural formulas but also to develop the ability to prove theorems and comprehend the relationships between geometric concepts. As the future of geometry education shifts towards emphasizing argumentation, proof, visualization, figuration, and instrumentation processes (Jones et al., 2024), these skills become even more critical. To meet these demands, pre-service mathematics teachers must achieve the level of formal deduction and rigor described in Van Hiele's theory (Fuys, 1988; Mariotti & Pedemonte, 2019), which involves mathematical proof, concept generalization, and understanding logical structures. This level supports logical reasoning, in-depth mathematical mastery, and the ability to teach complex geometric concepts effectively.

However, most pre-service mathematics teachers in Indonesia remain at the visualization and analysis levels (Alex & Mammen, 2018; Armah, 2024; Bleeker, 2011; Jones & Rodd, 2001) and the informal deduction (Armah, 2024; Mawarsari et al., 2023; Naufal et al., 2021; Watan & Sugiman, 2018) according to Van Hiele's model. At these levels, the focus is primarily on recognizing basic shapes and attributes without deep exploration of formal deduction or generalization. Additionally, research shows that some educators even skip or delay teaching geometry due to insufficient pedagogical knowledge (Niyukuri et al., 2020). As a result, assessment practices often focus on rote memorization rather than conceptual exploration, reducing opportunities for understanding and applying geometric concepts at deductive and rigorous levels (Vieira & de Costa Trindade Cyrino, 2022). This issue is further highlighted by (Mukuka & Alex, 2024), who reported that only 13% of pre-service mathematics teachers could solve geometry problems due to a lack of exposure to advanced-level geometry exercises. The assessments given to these teachers have primarily emphasized procedural application (Armah, 2024). These conditions hinder pre-service mathematics teachers's ability to perform the mathematical proofs required to solve advanced geometry problems (Alex & Mammen, 2018; Mukuka & Alex, 2024). This issue is increasingly urgent, given the importance of geometry in shaping logical and deductive thinking across disciplines, especially in mathematics (Clements & Battista, 1992; Hanna, 1995; Ramírez-Uclés & Ruiz-Hidalgo, 2022; Schoenfeld, 1994; Usiskin, 1982).

In response to this issue, many researchers suggest that geometry tasks in higher education should include elements that encourage formal deduction and rigor, such as theorem proving and the generalization of geometric concepts (Hanna & Sidoli, 2007; Jones & Rodd, 2001). According to research by (Buchbinder, 2020; Hanna, 2018; Hanna & de Villiers, 2021), pre-service mathematics teachers who are accustomed to proof-based tasks tend to develop stronger deductive reasoning skills, which are essential for solving more complex problems. Recent research by (Vieira & de Costa Trindade Cyrino, 2022) also shows that pre-service mathematics teachers who frequently engage with proof-based tasks have a deeper conceptual understanding compared to those who focus solely on procedural tasks. These findings emphasize the importance of incorporating formal proof exercises into the geometry curriculum to help pre-service mathematics teachers understand complex geometric relationships and hone their critical thinking skills.

However, many problems in analytic geometry courses are not designed to challenge pre-service mathematics teachers beyond informal deduction, limiting their opportunities to achieve formal deduction and rigor. Furthermore, there remains a gap in the literature (Bleeker, 2011) regarding how these problems can be redesigned to develop high-level geometric thinking skills in pre-service mathematics teachers (Armah, 2024).

This study addresses this gap by analyzing four analytic geometry problems given to pre-service mathematics teachers and providing specific recommendations to improve their design to encourage formal deduction and rigor (Ariawan et al., 2024; Breyfogle & Lynch, 2010). Unlike previous studies that merely emphasized the importance of developing deductive thinking (Maarif, Alyani, et al., 2020; Maarif, Wahyudin, et al., 2020), this research offers concrete steps and practical solutions to improve the existing problems. Recent contributions from Vieira & de Costa Trindade Cyrino, (2022) affirm the need to integrate proof-based learning into geometry education to enhance conceptual mastery and the importance of formal proofs in advancing geometric thinking (Fuys, 1988; Gutierrez et al., 2004). Building on these insights, this study provides a novel contribution to geometry teaching by proposing redesigns that can be directly applied to teaching analytic geometry in higher education. The focus of this research is to evaluate the extent to which current analytic geometry problems encourage pre-service mathematics teachers to achieve formal deduction and rigor and to offer specific recommendations for redesigning these problems to effectively develop high-order geometric thinking skills.

## Methods

### Research design

This study employs a case study design to analyze geometry problems assigned in an analytic geometry course at a university in South Sumatra, Indonesia. The case study approach was chosen to enable an in-depth examination of these problems, with a specific focus on how they assess the geometric thinking skills of pre-service mathematics teachers. The main objective of this study is to determine whether these problems evaluate the geometric thinking skills of pre-service mathematics teachers, particularly at the level of formal deduction and rigor within the Van Hiele model of geometric thinking. This design allows for a detailed investigation of the types of geometric thinking assessed by these problems, which might not be captured by broader quantitative methods. The research instruments and procedures were carefully designed to ensure that the analysis could reveal whether the assessments used adequately measure the desired higher-level geometric thinking skills. The Van Hiele model was chosen because it is a well-established framework for assessing the development of geometric thinking and is suitable for evaluating the complexity of problems used in analytic geometry courses.

## Participants

The subjects of this study are the analytic geometry problems provided in two different assessment contexts: exams and homework assignments from the analytic geometry course in the second semester of the 2023/2024 academic year. No human participants were directly involved, as the focus of this research is solely on the content analysis of the given problems and the written responses of the pre-service mathematics teachers.

## Research instruments

This study utilizes document analysis as the primary instrument, with analytic geometry problems serving as the data source. The geometry problems were collected from two sources: the mid-semester exam and homework assignments given during the analytic geometry course. All problems were gathered mid-semester to ensure a comprehensive analysis of the assessments provided in the course. The problems cover topics such as the equation of a straight line, Hesse normal form, and the position of a line relative to another line. A coding system based on the Van Hiele geometric thinking model, developed by (Mayberry, 1983) is used in this study to classify each question according to the level of geometric thinking assessed, ranging from visualization to rigor. This coding system is designed by adapting the geometry task indicators for each level of geometric thinking by (Fuys, 1988; Hohol, 2019; Usiskin, 1982), and has undergone validation by geometry experts, being deemed valid to evaluate whether the questions assess basic geometric thinking skills or promote higher-level geometric thinking abilities, such as formal deduction and rigor, as presented in Table 1.

**Table 1.** Coding of problems based on Van Hiele levels of geometric thinking

| Level | Description | Coding Indicators |
|---|---|---|
| Visualization (level 0) | Identifying geometric shapes based on visual observation without understanding their formal properties. | The problem only asks the pre-service mathematics teachers to recognize shapes without requiring an analysis of their properties |
| Analysis (Level 1) | Differentiating geometric shapes based on their properties, using inductive reasoning without deductive proof. | The problem asks the pre-service mathematics teachers to analyze the properties of shapes without requiring formal proof. |
| Informal Deduction (Level 2) | Understanding the relationships between geometric properties and reasoning with simple arguments, but not yet using formal proof. | The problem asks the pre-service mathematics teachers to understand relationships between geometric properties and use simple arguments. |
| Formal Deduction (Level 3) | Using deductive reasoning to prove geometric properties and solve geometry problems formally. | The problem asks the pre-service mathematics teachers to perform logical proofs to solve geometric problems. |
| Rigor (Level 4) | Generalizing and analyzing geometric structures, as well as comparing more complex geometric properties. | The problem asks the pre-service mathematics teachers to generalize or compare various geometric structures. |

The questions, which have been classified based on coding, were then redesigned according to the characteristics of questions that fall within the formal deduction or rigor level of the Van Hiele model. Subsequently, the redesigned geometry questions were validated by geometry experts using an expert validation sheet, referring to the indicators in the following Table 2, which was developed by the researcher after undergoing both theoretical validation and validation by mathematics education lecturers.

**Table 2.** Assessment aspects of the redesigned analytic geometry problems by geometry experts

| Aspects | Indicators | Descriptors | Relevant References |
|---|---|---|---|
| Learning Objectives | 1. Alignment of the problem with the learning objectives | The problem tests the ability for formal deduction and precision in analytic geometry | Emphasizes a deductive approach in solving mathematical problems (Polya, 1945) |
| | | | Provides an understanding of the development of geometric thinking at the formal level (Van Hiele, 1986). |
| | 2. The problem's ability to encourage abstract thinking | The problem encourages students to think more abstractly and generalize geometric concepts. | Emphasizes the importance of the deductive process and communication in mathematics, which can be used to foster abstract thinking in students (Sfard, 2008). |
| | 3. Relevance of the problem to basic geometric concepts | The problem is relevant for understanding basic geometric concepts such as gradients, line equations, normal length, and their applications. | Provides standards for teaching basic geometric concepts through a logical and deductive approach NCTM (2000). |
| Geometric Concept Alignment | 1. Validity of the geometric concepts used | The problem contains valid geometric concepts relevant to the topic, such as segment division, Hesse normal form, perpendicular lines, etc. | Provides a theory stating that understanding geometric concepts develops in more systematic stages, supporting the validity of the geometric concepts used in the problem (Van Hiele, 1986). |
| | 2. Valid deductive proof related to geometric formulas | The problem requires valid deductive proof related to the formulas or concepts used. | Mathematical proofs are essential in geometry education and should be performed deductively and logically (Hanna & de Villiers, 2021). |
| | 3. Connection between geometric concepts | The problem allows students to understand the deep relationships between points, lines, and other geometric properties within the problem's context. | Provides a mathematical basis that introduces connections between geometric concepts through deductive approaches and formal proofs (Epp, 2011). |
| Difficulty Level & Precision | 1. The challenge posed by the problem for students at an appropriate understanding level | The problem is sufficiently challenging and appropriate for the prospective mathematics teachers' understanding. | Teaches the importance of didactical design that creates challenges suitable for the students' level of understanding (Artigue, 2009). |
| | 2. Deep deduction, not just applying formulas | The problem tests students' understanding through deeper deductions rather than just applying formulas or standard procedures. | Emphasizes the importance of deduction in problem-solving, rather than merely applying formulas or procedures (Polya, 1945). |

| Aspects | Indicators | Descriptors | Relevant References |
|---|---|---|---|
| | 3. Precision in calculations and geometric analysis | The problem requires precision in performing calculations and analyzing geometric objects involved in the problem. | Discusses teaching standards involving precision in geometric analysis and the application of mathematical concepts NCTM (2000). |
| Relevance to Concept Application | 1. Analysis of changes in geometric contexts | The problem prompts students to analyze changes occurring in geometric contexts, such as changes in ratios or constants. | States that analyzing changes is part of the developing mathematical thinking process (Sfard, 2008). |
| | 2. Encouragement of exploration and generalization of geometric concepts | The problem encourages students to explore and generalize geometric concepts in more general situations. | Explains how students can develop through exploration and generalization of geometric concepts (Van Hiele, 1986). |
| | 3. Comparison of different geometric cases | The problem creates opportunities for students to compare different geometric cases and identify patterns or general conclusions. | Teaches how to compare different cases to derive general patterns or principles in geometry (Polya, 1945). |
| Proof Validity & Rigor | 1. Rigorous and logical proof | The problem leads to rigorous and logical proofs, following systematic steps to reach valid conclusions. | Emphasizes the importance of logical and rigorous proofs in mathematics education, which must be achieved in geometry problems (Hanna & de Villiers, 2008). |
| | 2. Clarity and systematic approach in proofs | The proof in the problem is clear and easy to understand, following a logical sequence, and free from ambiguity or confusion. | Teaches the importance of a clear and systematic approach in mathematical proofs within didactical contexts (Artigue, 2009). |
| | 3. Precision in terminology and definitions | The problem uses precise and appropriate terminology and definitions in the context of analytic geometry, avoiding confusion or ambiguity. | Highlights the importance of using correct terminology to build clear and formal understanding of geometry (Van Hiele, 1986). |
| Alignment with Educational Objectives | 1. Development of higher-order mathematical thinking skills | The problem develops higher-order mathematical thinking skills, such as analysis, synthesis, and generalization, in the context of analytic geometry. | Discusses the importance of developing higher-order mathematical thinking skills in mathematics education NCTM (2000). |
| | 2. Preparation of prospective teachers to teach geometric concepts | The problem supports the goal of preparing prospective mathematics teachers to understand and teach more complex geometric concepts in the future. | States that good mathematics education must develop communication and analytical skills to teach geometry effectively (Sfard, 2008). |

### Data analysis techniques

The data were analyzed using content analysis techniques, focusing on determining the levels of geometric thinking assessed by each problem. The analysis was conducted in several steps, as presented in Table 3.

**Table 3.** Evaluation criteria for the analytic geometry problems used

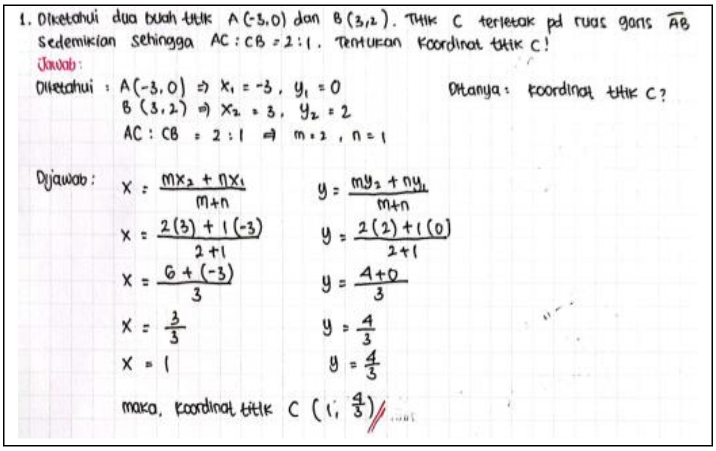
| Evaluation Criteria | Description |
|---|---|
| Classification of problems based on Van Hiele levels of geometric thinking ((Hohol, 2019; Mayberry, 1983; Usiskin, 1982) | Each problem is classified according to the Van Hiele levels, ranging from visualization, analysis, informal deduction, formal deduction, to rigor. Problems are evaluated to determine whether they encourage pre-service mathematics teachers to use basic geometric reasoning or require them to employ higher-order geometric thinking, such as proving geometric properties or making generalizations (rigor). |
| Validation with geometric concepts and expert | - **Comparison with Established Geometric Concepts**: The classification results were compared with recognized geometric principles and frameworks to verify accuracy.<br>- **Expert Review**: The classification and evaluation process was then reviewed by experts in mathematics education and geometry to ensure consistency and alignment with theoretical foundations, referring to Table 2.<br>- **Iterative Refinement**: Problems that did not reach the levels of formal deduction or rigor were revisited and refined based on feedback to improve their ability to assess higher-order geometric thinking. |
| Recommendations for Problem Redesign | Problems that did not effectively encourage formal deduction or rigor were given detailed recommendations for redesign to better align with the goals of developing higher-level geometric thinking skills. |

## Results

### Classification of problems based on levels of geometric thinking

Each problem is classified according to the Van Hiele levels, ranging from visualization, analysis, informal deduction, formal deduction, to rigor. The problems are evaluated to determine whether the lecturer encourages pre-service mathematics teachers to use basic geometric reasoning or requires them to engage in higher-order geometric thinking, such as proving geometric properties or making generalizations (rigor). The lecturer used a total of 4 problems, covering topics such as the equation of a straight line, Hesse normal form, and the position of a line relative to another line. The following are the evaluation results provided by the researchers for each problem, followed by the classification of each problem based on the thinking level presented in Table 1. Below are the 4 problems given by the lecturer along with a sample of one correct pre-service mathematics teachers response.

**Problem 1: Determining the coordinates pf point $C$ on line segment $AB$**


1. Diketahui dua buah titik A(-3,0) dan B(3,2). Titik C terletak pd ruas garis $\overline{AB}$ sedemikian sehingga AC : CB = 2 : 1. Tentukan koordinat titik C!

Jawab:

Diketahui : $A(-3,0) \Rightarrow x_1 = -3,\ y_1 = 0$

$B(3,2) \Rightarrow x_2 = 3,\ y_2 = 2$

$AC : CB = 2 : 1 \Rightarrow m = 2,\ n = 1$

Ditanya : koordinat titik C?

Dijawab :

$x = \frac{mx_2 + nx_1}{m+n}$ $\quad y = \frac{my_2 + ny_1}{m+n}$

$x = \frac{2(3) + 1(-3)}{2+1}$ $\quad y = \frac{2(2) + 1(0)}{2+1}$

$x = \frac{6 + (-3)}{3}$ $\quad y = \frac{4+0}{3}$

$x = \frac{3}{3}$ $\quad y = \frac{4}{3}$

$x = 1$ $\quad y = \frac{4}{3}$

maka, koordinat titik C $(1, \frac{4}{3})$//

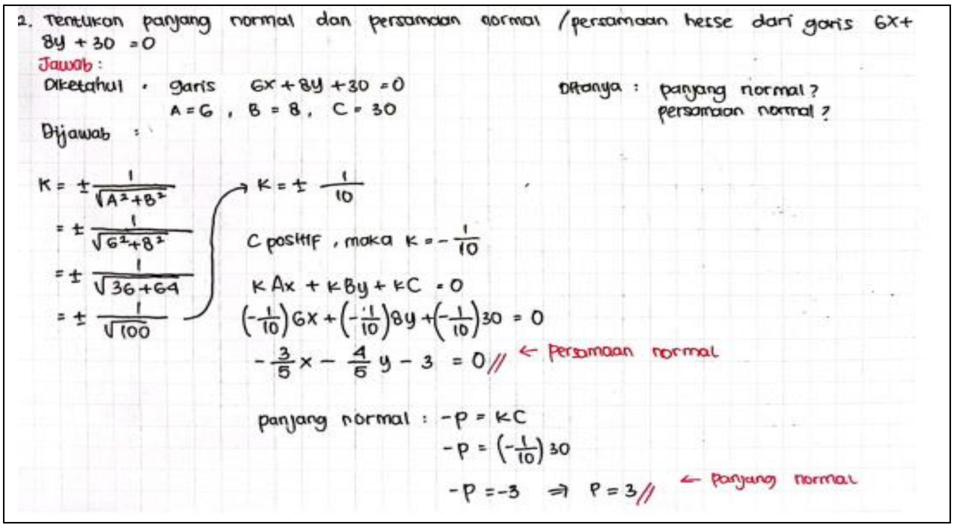
**Figure 1.** Problem 1 and Answer from A

Translation Problem 1:

Given two points $A(-3,0)$ and $B(3,-2)$, point $C$ lies on the line segment $\overline{AB}$ such that $AC{:}\,CB = 2{:}\,1$. Determine the coordinates of point $C$.

This problem asks pre-service mathematics teachers to calculate the coordinates of point C, which lies on the line segment $\overline{AB}$ with a ratio of segment lengths $AC{:}\,CB = 2 : 1$. This involves an understanding of the concept of coordinate ratios on a straight line and the application of the segment division formula in the coordinate plane. The pre-service mathematics teacher uses the segment division formula to calculate the coordinates of point $C$. The calculation is performed systematically by substituting the values $m$ = 2 and $n$ = 1 into the formula, then computing the $x$ and $y$ coordinates of point $C$. Since Problem 1 only requires pre-service mathematics teachers to understand the relationship between the lengths of line segments, the solution involves simply applying the given formula without requiring proof or generalization of the concept. This problem does not require formal proof and only uses simple reasoning; thus, it is categorized as informal deduction.

**Problem 2: Length of the normal and hesse normal form**


2. Tentukan panjang normal dan persamaan normal / persamaan hesse dari garis 6x + 8y + 30 = 0

Jawab:

Diketahui : garis $6x + 8y + 30 = 0$

$A = 6,\ B = 8,\ C = 30$

Ditanya : panjang normal? persamaan normal?

Dijawab :

$K = \pm \frac{1}{\sqrt{A^2+B^2}} = \pm \frac{1}{\sqrt{6^2+8^2}} = \pm \frac{1}{\sqrt{36+64}} = \pm \frac{1}{\sqrt{100}}$

$\rightarrow K = \pm \frac{1}{10}$

C positif, maka $K = -\frac{1}{10}$

$KAx + KBy + KC = 0$

$(-\frac{1}{10})6x + (-\frac{1}{10})8y + (-\frac{1}{10})30 = 0$

$-\frac{3}{5}x - \frac{4}{5}y - 3 = 0$// ← Persamaan normal

panjang normal : $-p = KC$

$-p = (-\frac{1}{10})30$

$-p = -3 \Rightarrow p = 3$// ← panjang normal

**Figure 2.** Problem 2 and answer from A

Translation Problem 2:

Determine the normal length and the Hesse normal form of the line $6x + 8y + 30 = 0$.

This problem asks pre-service mathematics teachers to calculate the length of the normal and write the Hesse normal form equation for the line $6x + 8y + 30 = 0$. The pre-service mathematics teachers are only required to apply the formula to calculate the length of the normal and to construct the Hesse normal form equation. This problem focuses on the application of formulas without requiring formal proof or generalization of the concepts used. Therefore, this problem is categorized at the analysis level.

**Problem 3: Perpendicular line equation**

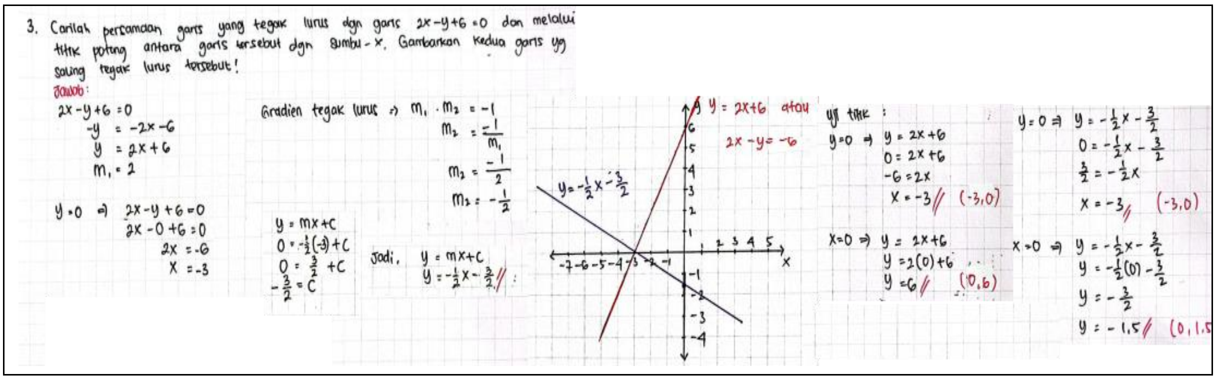



**Figure 3.** Problem 3 and answer from A

Translation Problem 3:

Find the equation of the line that is perpendicular to the line $2x - y + 6 = 0$ and passes through the intersection of the line with the x-axis. Draw both perpendicular lines.

This problem asks pre-service mathematics teachers to find the equation of a line perpendicular to the line $2x - y + 6 = 0$ and to graph both lines on a Cartesian plane. The pre-service mathematics teachers are required to use the gradient formula and analyze the relationship between two perpendicular lines. They were also able to accurately graph the lines on the coordinate plane, demonstrating their visualization skills. Although the pre-service mathematics teachers were able to solve the problem correctly, the problem did not require a formal proof of why the gradient of perpendicular lines is the negative reciprocal. Therefore, this problem is categorized as informal deduction.

**Problem 4: Tangent line with circle**

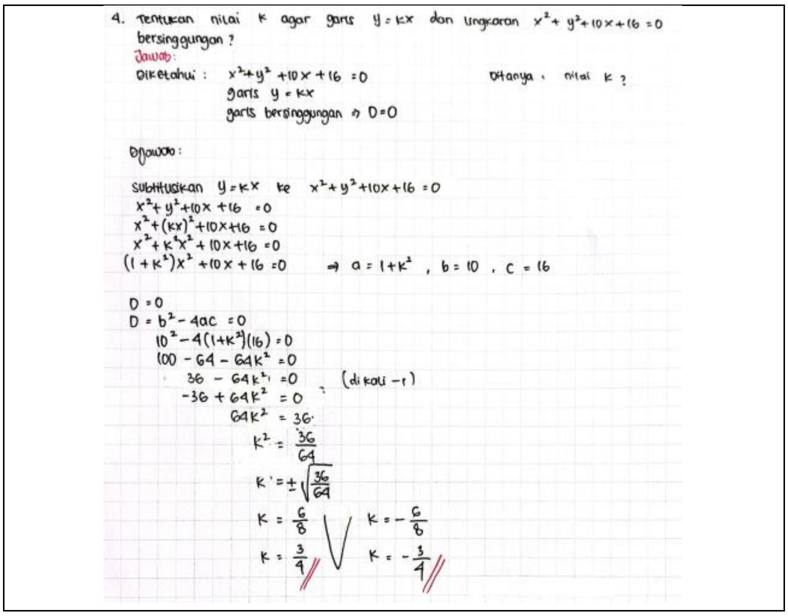



**Figure 4.** Problem 4 and answer from A

Translation Problem 4:

Determine the value of $k$ so that the line $y = kx$ is tangent to the circle lingkaran $x^2 + y^2 + 10x + 16 = 0$.

This problem asks pre-service mathematics teachers to determine the value of $k$ so that the line $y = kx$ is tangent to the circle $x^2 + y^2 + 10x + 16 = 0$. the problem requires pre-service mathematics teachers to use the discriminant from a quadratic equation without formally proving the properties of the discriminant and how the tangency occurs. The problem focuses on the application of formulas and algebraic steps; thus, it is categorized at the analysis level and informal deduction.

**Table 4.** Classification of problems based on levels of geometric thinking

| No. | Problem | Level of van Hiele | Description |
|---|---|---|---|
| 1 | Coordinates of point C on line segment $\overline{AB}$ | Informal Deduction | Pre-service mathematics teachers understand the relationship between line segments and apply a formula without formal proof. |
| 2 | Length of the Normal and Hesse Normal Equation | Analysis | The problem only requires the application of formulas to calculate the length of the normal and construct the equation. |
| 3 | Equation of Perpendicular Lines | Informal Deduction | Pre-service mathematics teachers use the gradient formula to solve the problem without formal proof. |
| 4 | Tangent of the Line $y = kx$ with the Circle | Analysis, Informal Deduction | The problem focuses on the application of the discriminant without formal proof or generalization. |

Based on Table 4, it is evident that the problems used by the lecturer in the analytic geometry course only reach the levels of analysis and informal deduction in the Van Hiele model. These problems do not require pre-service mathematics teachers to reach the levels of formal deduction and rigor, which are the highest levels of geometric thinking. Although the problems demand that pre-service mathematics teachers use formulas and understand basic relationships between geometric elements, they are not yet required to perform formal proofs or generalizations of the concepts used.

### Validation with geometric concepts

Since the problems used by the lecturer do not reach the level of geometric thinking that should be required in an analytic geometry course from visualization to reaching the level of formal deduction or rigor, a further review was conducted to identify potential improvements, and recommendations were provided for redesigning the problems to better assess higher-level geometric thinking. The detailed evaluation results, issues, and recommendations for redesigning the problems to more effectively measure higher-order geometric thinking, such as formal deduction and rigor, are presented in Table 5.

**Table 5.** Evaluation of problems based on formal deduction and rigor levels

| | | |
|---|---|---|
| **Problem 1:** Coordinates of point C on line segment $\overline{AB}$ | | |
| **Geometric Principle Applied** | **Division of a line segment** | |
| Redesign Recommendation | Formal Deduction | Prove why the line segment division formula holds. Instead of just calculating point C, pre-service mathematics teachers should be asked to formally demonstrate how this formula is derived based on geometric principles. |
| | Rigor | Generalize from the given situation by analyzing cases where the ratio between $AC$ and $CB$ is not constant, or if point C is located outside line $AB$. |
| **Problem 2:** Length of the Normal and Hesse Normal Form | | |
| Geometric Principle Applied | Hesse normal form, length of the normal | |
| Redesign Recommendation | Formal Deduction | Logically prove why the Hesse normal formula is valid. |
| | Rigor | Compare different representations of a line, such as the general form, slope-intercept form, and Hesse normal form. |
| **Problem 3:** Equation of a perpendicular line | | |
| **Geometric Principle Applied** | **Gradient for perpendicular lines** | |
| Redesign Recommendation | Formal Deduction | Deductively prove the property of perpendicular lines. This proof should be based on logical arguments using the properties of the gradient and the equation of a line. |
| | Rigor | Compare the equations of parallel and perpendicular lines in various forms. Pre-service mathematics teachers should compare how parallel and perpendicular lines are represented in general form and slope-intercept form. |

| **Problem 4:** Tangency of line $\boldsymbol{y = kx}$ with a circle | | |
|---|---|---|
| **Geometric Principle Applied** | **Discriminant to determine the tangent line to a circle** | |
| Redesign Recommendation | Formal Deduction | Prove that a zero discriminant ensures the line is tangent to the circle. Instead of just using the formula, pre-service mathematics teachers should logically show how the discriminant works in the context of quadratic equations and the relationship between the line and the circle. |
| | Rigor | Analyze how changes in the constant $c$ affect the conditions for tangency. Compare how lines with different gradients interact with the circle in various situations. Examine the differences and similarities between lines that are tangent, parallel, and perpendicular to the circle, which requires an in-depth analysis of geometric properties. |

Based on the recommendations outlined in Table 5, the following are the improvements made to the problems to reach the level of formal deduction or rigor.

**Table 6**. Improvements to analytic geometry problems at the level of formal deduction or rigor

| **Problem** | **Redesigned Problem** |
|---|---|
| Problem 1 | Given the coordinates of points $A$ (−3,0) and $B$ (3,2):<br>- Determine the coordinates of point $C$ hat divides line segment $AB$ in the ratio $AC$:$CB$=2:1.<br>- Deductively prove why the segment division formula $x = \frac{mx_2+nx_1}{m+n}$ holds in this case.<br>- Analyze how the position of point $C$ changes if the ratio AC:CB becomes k: 1, where $k$ is arbitrary. Generalize the result.<br>- Prove whether this segment devision concepts applies if point $C$ is outside of the line segment AB.<br>- Compare how changing the ratio AC:CB affects the gradient of line $AB$ and other geometric properties. |
| Problem 2 | A line has the equation $6x + 8y + 30 = 0$:<br>- Calculate the length of the normal and write the Hesse normal form equation for the line.<br>- Deductively prove why the Hesse normal formula $\frac{A_x+B_y+C}{\sqrt{A^2+B^2}}$ holds in this case. Explain each step of the proof.<br>- Consider for this line $6x + 8y + C = 0$. Construct the Hesse Normal form equation for this line and generalize how cahnges in the constant C affect the length of the normal. Analyze if there is a discernible patterm from these results. |
| Problem 3 | - Find the equation of the line perpendicular to $2x - y + 6 = 0$ that passes through the intersection point of this line with the x-axis.<br>- Prove that two perpendicular lines have gradients that are negative reciprocals of each other. Use the line equation $2x - y + 6 = 0$ and its perpendicular counterpart to support your proof.<br>- Compare the properties of the line $2x - y + 6 = 0$ with another line that is parallel to it, and prove whether there is a general equation for parallel and |

| Problem | | Redesigned Problem |
|---|---|---|
| | | perpendicular lines. Generalize the relationship between two perpendicular and parallel lines in various line equation forms (general form and slope-intercept form). |
| Problem 4 | - | Prove that the condition for the line $y = kx$ bto be tangent to the circle $x^2 + y^2 + 10x + 16 = 0$ is when the discriminant $D = 0$. JExplain the steps of your proof in detail. |
| | - | If the equation of the circle is changed to $x^2 + y^2 + 10x + c = 0$ etermine how changes in the constant c affect the value of $k$ so that the line $y$=$kx$ remains tangent to the circle. Prove this generalization. |

**Validation by geometry experts**

Table 7 below shows the results of expert validation conducted by geometry lecturers in the mathematics education program at a public university in South Sumatra. The expert validation results indicate excellent alignment between each problem presented and the learning objectives, relevant geometric concepts, difficulty level and precision, applicability of the concepts, proof validity and rigor, as well as alignment with educational objectives. Each validated problem meets these criteria well, suggesting that the problems are designed to support in-depth understanding and the development of geometric skills among prospective mathematics teachers.

**Table 7.** Results of expert checklist validation on redesigned geometric analytic problems

| Problem | Learning Objectives | Geometric Concept Alignment | Difficulty Level & Precision | Relevance to Concept Application | Formal Deductive & Rigor | Alignment with Educational Objectives |
|---|---|---|---|---|---|---|
| Problem 1 | √ | √ | √ | √ | √ | √ |
| Problem 2 | √ | √ | √ | √ | √ | √ |
| Problem 3 | √ | √ | √ | √ | √ | √ |
| Problem 4 | √ | √ | √ | √ | √ | √ |

According to the evaluation from the experts, the redesign of these analytic geometry problems has proven to be highly effective in enhancing formal deduction and rigor skills. Each revised problem requires students not only to apply mathematical formulas but also to prove geometric relationships through a formal deductive process. This revision provides an opportunity for prospective mathematics teachers to refine their ability to use logical deduction and develop a more systematic and rigorous approach to geometric thinking.

## Discussion

This study's findings indicate that the analytic geometry problems given to pre-service mathematics teachers tend to emphasize analytical thinking and informal deduction, without pushing students to reach the level of formal deduction and rigor as described in the Van Hiele model. These findings align with previous research, which suggests that geometric thinking in higher education is often limited to procedural levels, providing fewer opportunities for students to develop deductive skills and engage in formal proofs (Fuys, 1988).

In Problem 1, students were asked to determine the coordinates of point C, which divides line segment AB in a given ratio. This problem required the application of the segment division formula, a basic concept in analytic geometry that heavily relies on the Cartesian coordinate system (Arista Rizki, 2018). The analysis showed that students were able to perform the correct calculations, but they were not asked to prove why the formula works. This resulted in the problem remaining at the informal deduction level. According to (Armah, 2024), geometry problems in higher education often focus on the mechanical application of formulas, rather than on proof or deeper conceptual understanding. (Fuys, 1988) explains that students who are not engaged in formal proof tasks tend to remain at the informal deduction level, as they are not given the opportunity to develop deeper logical reasoning skills. This result is further supported by (Hanna & de Villiers, 2021; Hanna & Sidoli, 2007; Maarif, Wahyudin, et al., 2020), who emphasize the importance of formal proof in enhancing advanced geometric thinking.

To improve the quality of this problem, a formal proof component could be added, for example, by asking pre-service mathematics teachers to explain why the segment division formula holds based on the properties of lines in a coordinate plane. Additionally, the problem could be expanded to include generalization, such as asking students to investigate what happens when the ratio changes dynamically. This would encourage students to reach the rigor level, where they must understand how geometric relationships apply in more general cases. This approach aligns with the primary goals of geometry education in schools, which should be mastered and emphasized for pre-service mathematics teachers, such as developing logical thinking skills, teaching how to read and interpret mathematical arguments (Clements, 1996). Furthermore, for students to be able to solve geometric problems, learning activities must be more rigorous compared to simply learning facts and memorization (Ruseffendi, 1988).

In Problem 2, students were asked to calculate the length of the normal and write the Hesse normal form equation of a line. This problem primarily assesses procedural knowledge related to linear equations in the Cartesian coordinate system. As explained by (Arista Rizki, 2018), calculating distances and line equations in normal form are fundamental concepts in analytic geometry. However, the problem does not require students to prove why the Hesse normal form is valid or to compare it with other linear equation forms, such as the slope-intercept or general form. As a result, this problem remains at the analytical level, consistent with the findings of (Ramírez-Uclés & Ruiz-Hidalgo, 2022), who found that university tasks often only demand procedural accuracy without involving proof or conceptual exploration. This represents a missed opportunity because students were not asked to generalize, such as exploring how changes in the values of the equation affect the result. If they were asked to prove the Hesse formula or generalize the equation, they could progress to the rigor level, as suggested by (Alex & Mammen, 2018; Bleeker, 2011; Jones & Rodd, 2001).

To improve this problem, students could be asked to prove the validity of the Hesse normal form equation, for example, by explaining how the equation is derived from the fundamental properties of lines in the coordinate system. They could also be tasked with comparing different forms of line equations and determining in which situations each form is more appropriate. This would push geometric thinking to a higher level.

In problem 3, which involves finding the equation of a perpendicular line, a similar issue was found. Students were able to calculate the correct gradient for the perpendicular line, but they were not asked to prove why the gradients of perpendicular lines must be negative reciprocals of each other. According to (Arista Rizki, 2018), understanding the relationship between the gradients of intersecting lines is fundamental in analytic geometry, but requiring formal proof of this property would push students to the deduction stage in the Van Hiele model. This result aligns with the findings of (Ramírez-Uclés & Ruiz-Hidalgo, 2022), which noted that many geometry tasks in higher education fail to challenge students to engage in formal reasoning. This problem could be improved by asking students to prove the relationship between the gradients of two perpendicular lines using the definition of the scalar product or the properties of angles in the coordinate plane. Students could also be challenged to compare parallel lines, perpendicular lines, and other types of lines in terms of their geometric properties. This would help them develop a more general understanding of the relationships between lines in various cases.

Lastly, in Problem 4, students used the discriminant to determine whether the line is tangent to the circle. However, they were not asked to prove why the discriminant method works in this context or to explore how changes in the equation affect the conditions for tangency. As explained by (Arista Rizki, 2018), the relationship between lines and conic sections is an important area in analytic geometry, but without requiring formal proof or exploring general cases, this problem remains at the level of informal deduction. (Alex & Mammen, 2018) also found that many geometry tasks fail to promote deep conceptual understanding or formal reasoning. (Hanna & Sidoli, 2007) further emphasized that university-level problems should include a proof component, encouraging students to think more deeply about the relationship between algebraic equations and geometric properties.

To improve the quality of this problem and move it toward the rigor level, students could be asked to derive the discriminant condition for tangency from basic principles, using the properties of conic sections and lines. They could also be encouraged to explore how changes in the parameters of the circle or line affect the conditions for tangency, helping them understand the more general relationship between algebraic equations and geometric properties.

Based on the analysis and evaluation of the four problems used in the analytic geometry course, where the students are pre-service mathematics teachers who will eventually teach geometry and broader mathematics content at the secondary school level, if the emphasis remains solely on procedural application without deeper engagement in formal proof, the goals of studying geometry at the university level as future teachers will never be fully achieved. These findings support the conclusion that university-level geometry assessments, at least in this context, tend to emphasize procedural knowledge over conceptual understanding. This aligns with previous studies that have found a similar gap between the procedural focus in geometry problems and the need to develop higher-order thinking skills through proof and generalization tasks (Armah, 2024; Mawarsari et al., 2023; Naufal et al., 2021). It also shows that introducing tasks that engage students in formal deduction and generalization is crucial for improving their understanding of geometry (Scristia et al., 2021, 2022; Sumarni et al., 2020). Studying geometry means that students will develop the ability to understand geometric concepts, reason geometrically, represent geometric ideas, communicate geometric concepts, solve geometric problems, and think geometrically. Students who learn geometry will understand geometric shapes and structures and be able to analyze the characteristics and relationships between geometric structures, develop reasoning and justification skills, which ultimately lead to proof (Jones & Tzekaki, 2016; Ramírez-Uclés & Ruiz-Hidalgo, 2022). Geometry learning is essential for building mathematical reasoning that enables students to see connections between geometric objects, identify patterns, and provide logical justification for geometric conjectures (G. Stylianides et al., 2017). (Reid & Knipping, 2019) also revealed that in geometry teaching, there is a greater focus on proof, using geometric proof as a tool to understand fundamental mathematical concepts.

Therefore, to achieve higher levels of geometric thinking, analytic geometry tasks need to be redesigned to incorporate formal proof and generalization. The results of expert validation show that the redesigned analytic geometry problems are effective in enhancing formal deduction and rigor skills among prospective mathematics teachers. Each problem not only requires students to apply formulas but also to prove geometric relationships through a formal deductive process, which develops logical deduction abilities and more systematic and rigorous geometric thinking. This validation also emphasizes the importance of integrating reasoning and proof into the geometry curriculum, which has been shown to deepen the understanding of geometry (Stylianides G, 2008; Weingarden & Buchbinder, 2023). Therefore, the revised problems support the development of higher-level geometric thinking skills, which will enhance the quality of mathematics teaching. This study shows that educators need to include tasks that challenge students to justify their solutions and engage in deeper conceptual thinking. Future research could explore how these redesigned tasks affect students' deductive thinking development in the long term. Additionally, further studies could assess the impact of implementing these tasks in a broader educational context, both at the secondary school and university levels.

## Conclusion

This study aims to analyze whether the geometry problems given in an analytic geometry course assess students' geometric thinking abilities, particularly at the levels of formal deduction and rigor as outlined in the Van Hiele model. Based on the research findings, it was discovered that these problems primarily assess analytical and informal deductive skills, with little emphasis on requiring students to engage in proof or generalization. These problems tend to encourage the application of familiar formulas without necessitating deeper logical reasoning or formal proof, limiting students' ability to develop advanced geometric thinking.

This research shows that although students are able to apply formulas and solve problems correctly, the absence of tasks requiring formal deduction and rigorous generalization hinders their development in achieving advanced geometric thinking. This highlights a gap in the assessment design, which could be addressed by incorporating more tasks that encourage formal proof, logical reasoning, and comparisons of geometric structures.

The implications of this study suggest that, in order to support the development of higher-level geometric thinking, specifically formal deduction and rigor, it is essential to redesign geometry problems by incorporating elements of proof, generalization, and comparative analysis, all of which lead to reasoning processes. Such tasks not only enhance students' understanding of geometric concepts but also better prepare them for more complex mathematical reasoning.

However, this study has limitations, as it only analyzes a small number of problems from one course. Future research could expand the scope by including findings from the learning processes conducted by lecturers during the course, to determine whether the instruction has adequately prepared students for reasoning and logical argumentation. Further studies could also analyze a wider variety of assessments conducted by lecturers during the learning process to provide a more comprehensive understanding of the extent to which geometry assessments promote higher-order thinking.

Overall, this study highlights the need for changes in the design of geometry problems to ensure that pre-service mathematics teachers not only assess procedural knowledge but also develop deeper and more sophisticated geometric thinking. By addressing this gap, educators can better support students in reaching their full potential in mathematical thinking.

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**Conflict of Interest**

The authors affirm that there are no conflicts of interest related to the publication of this manuscript. Furthermore, ethical issues such as plagiarism, misconduct, data fabrication or falsification, duplicate publication or submission, and redundancies have been addressed and resolved by the authors.

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**Author Contributions**

**Scristia**: Developed the concept, formulated the idea, wrote the initial draft, created the visualization, collected data, conducted formal analysis, and designed the methodology; **Tatang Herman:** Contributed to concept development, writing, and reviewing, and was responsible for validation and supervision; **Septy Sari Yukans:** Involved in writing, reviewing, and data collection.

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