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Students' conceptual understanding of limit of functions reviewed from mathematical beliefs

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Abstract

Conceptual understanding is a student's cognitive structure characterised by the ability to transform and explain concepts in solving problems. Many students were unable to explain the concepts and the relationships between concepts in solving the limit of functions. This study aimed to explore students' conceptual understanding of the limit of functions in terms of mathematical beliefs. The subjects were 30 mathematics education students at Syiah Kuala University who had taken calculus courses for advanced real analyses. Data were collected using questionnaires, tests, and interviews. Data processing was carried out by reducing data, presenting, analysing, and drawing conclusions. The results of the study showed that students with strong mathematical beliefs demonstrated a more complete and integrated conceptual understanding of the limits of functions, as they could connect concepts, procedures, and graphical representations. In contrast, students with medium and low mathematical beliefs tended to focus only on procedural knowledge, often failing to explain underlying concepts or make meaningful connections between concepts and problem-solving steps. Based on the results, calculus lecturers need to build a strong conception of the material on the real number system and real functions so that the concept of the limit of functions is easily understood and memorised.

Keywords: conceptual understanding; limit of functions; mathematics belief

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Introduction

The limit of a function is one of the basic concepts for understanding mathematics, such as calculus and real analysis. Nurdin et al. (2019) stated that the limit of a function is an explanatory concept in the calculus course and is further discussed in the real analysis course. In addition, Oktaviyanthi et al. (2018) stated that limit is a basic concept in calculus, and that learning and teaching the limit of function in calculus is a topic to be studied in mathematics education. Thus, the limit of a function is one of the concepts that students need to learn and master well to understand advanced mathematical concepts.

Conceptual understanding is an important competency in learning and studying calculus. Faizah (2019) explained that conceptual understanding includes restating concepts, providing examples and non-examples, and applying concepts to solve problems. The Content Standards of Teacher Programs (NCTM, 2020) in the initial mathematics teacher preparation program recommends that the goal of studying calculus concepts for prospective teachers is to be able to demonstrate a conceptual understanding of limits, continuity, differentiation, and integration, and have a thorough background in the techniques and applications of calculus.

Several studies on students' understanding of the limit of functions have not shown encouraging results. Beynon and Zollman (2015) found that students are generally able to solve limit of functions problems using procedures but are unable to explain the reasons for using these procedures. A'zima et al. (2019) stated that high school students' understanding of the ability indicators for stating necessary and sufficient conditions, selecting and using procedures for solving limit of function is still low. Wabula and Cahyono (2017) found that students' understanding of determining the limit of a function by simply substituting from the left. The findings of several studies indicate that understanding the concept of the limit of a function is still a problem that must be addressed immediately.

Several studies on the understanding of the limit of function have been conducted extensively. For instance, Nurdin et al. (2019), Arsyad et al. (2021) investigated students' factual understanding of the formal definition of the limit of a function (and). Their findings indicate that research focusing on students' conceptual understanding, particularly in solving limits at one point and at infinity, remains limited. Similarly, Feudel and Biehler (2022) examined the interpretation of derivative functions, which are closely related but do not directly address the conceptual understanding of limits. Other studies have explored this topic. Winarso and Toheri (2017) analysed high school students' misconceptions in learning the concept of limits. Saleh (2018) studied students' understanding of limits based on their mathematical abilities, while Sulastri et al. (2022) described conceptual variations in understanding the limit at infinity at the university level. Thus, the conceptual understanding of the limit of a function remains an area that has not been widely investigated in mathematics education research.

Understanding mathematical concepts is not only about students being able to remember concepts, properties, and procedures, but also about being able to do various things by explaining, interpreting, and using facts, concepts, and principles in solving problems (Usman et al., 2017). Wabula and Cahyono (2017) explained that conceptual understanding is a person's ability to restate concepts, identify and explain, build connections, and reinforce them. While Klau et al. (2020) explained that conceptual understanding is a person's ability that is characterized by the ability to explain, identify, and make connections. So conceptual understanding is a person's ability that is characterized by the ability to explain concepts and relate them to the procedure for solving function limit problems, and explaining the relationship between the two.

Students' mathematical beliefs significantly influence their understanding of solving mathematical problems. These belief systems play a crucial role in shaping both teachers' and students' performance in addressing mathematical tasks. According to Usman et al. (2020), there is a strong relationship between mathematical beliefs and conceptual understanding. Their findings suggest that mathematical beliefs directly impact how individuals comprehend and approach problem-solving. Students with high, medium, or low belief levels exhibit distinct patterns of understanding when engaging in mathematical reasoning. Therefore, variations in students' mathematical beliefs can result in different levels of conceptual understanding, particularly in complex topics such as the limit of a function.

Belief is an assumption, attitude, or the result of a person's mental construction when solving a problem. Belief is the result of mental construction from codified experience, behaviour, and understanding in the problem-solving process (Ozturk and Guven, 2016). Minarni et al. (2018) stated that belief is an individual's basic mental assumption regarding understanding. Muhtarom et al (2018) describes mathematical beliefs as a tool of facts, rules, and skills. Angel-Cuervo et al. (2024) stated that mathematical beliefs are an accumulation of facts, rules, and skills used to achieve goals. Mathematical beliefs are a person's mental assumptions regarding understanding. Thus, beliefs are the result of mental construction in the form of the nature of reality, experience, behaviour, and understanding in solving a problem (Usman et al., 2020).

A student's mathematical belief about content is a belief about how mathematical truth and its proof are. The source of belief is how mathematical truth and validity are determined. The content beliefs that underlie the concept of limit in this study include beliefs about real numbers, infinite numbers, real functions, and how to state the assumption of their truth. The source of mathematical belief in this study is the determination of mathematical truth and validity. This study explores aspects of students' conceptual understanding of the limit of a function that are influenced by their mathematical beliefs.

This study presents a novel contribution through its integrated analysis of students' conceptual understanding of the limits of functions from the perspective of mathematical beliefs, specifically by distinguishing students' comprehension at a point and at infinity. Unlike previous studies that have tended to explore either conceptual understanding or mathematical beliefs in isolation, this study systematically connects both constructs in a comparative framework. This study examined students across varying levels of mathematical belief: high, medium, and low. This study aimed to identify patterns or models of students' conceptual understanding of limits. The analysis focused on five key aspects: the informal definition of limit, the formal definition, representation of the formal definition, explanation using examples and non-examples, and the ability to construct proofs of candidate limits. The expected outcome of this study is to reveal meaningful differences in conceptual understanding based on belief levels, thereby providing a nuanced view of how mathematical beliefs shape students' mastery of one of the most conceptually challenging areas of calculus.

Methods

Thus research was a qualitative study employing an exploratory approach. A qualitative method was chosen because it allowed the researcher to explore and interpret students' conceptual understanding of the limit of a function and its relationship with mathematical beliefs in depth.

The participants were 30 sixth-semester undergraduate students from the Mathematics Education Department at Syiah Kuala University, Banda Aceh. The selection criteria required that the students had completed calculus courses, including advanced topics such as limits and real analysis. From the 30 participants, six students were purposively selected for in-depth interviews. The selection was based on the results of the mathematical belief (MB) questionnaire, which categorised students into three levels of belief: low, middle, and high. Two students from each belief category were selected to represent a range of belief levels and to ensure variation in the analysis.

The instruments used in this study were divided into two types:

1. Main Instrument
   The main instrument was the researcher himself, acting as the data collector and interpreter, in line with qualitative research tradition (Yoon and Uliassi, 2022).
2. Supporting Instruments
   - Mathematical Belief Questionnaire (Dündar, 2015)
   - Conceptual Understanding Test on limit of functions, developed with reference to Usman et al. (2020) and the conceptual indicators presented in Table 1 below.

Table 1. Conceptual understanding indicators of the limit of a function

No.	Conceptual Understanding	Description
1	The limit of a function at one point if the function is given in the form of an equation	Able to determine limit of function by using concepts, principles, properties (theorems), methods, procedures, and explaining the relationship between concepts/principles and solution procedures.
2	The limit of a function is at one point if the function is given in graphical form	Able to determine limit of function by using concepts, principles, methods, procedures, and explaining concepts/principles with solution procedures.
3	The limit of a function at infinity if the function is given in equation form	Able to determine limit at infinity by using concepts, principles, methods, procedures, and explaining the relationship between concepts and solution procedures.
4	The limit at infinity if the function is given in graphical form	Able to determine limit at infinity by using concepts, principles, methods, procedures, and explaining the relationship between concepts and solution procedures.

- Interview Guidelines, designed with open-ended questions to explore the students' conceptual reasoning.

All instruments were validated through expert judgment involving three mathematics education lecturers who assessed the content and structure. Revisions were made according to their feedback to ensure alignment with the research objectives.

The research procedure was conducted in the following stages:

1. Preparation stage
   - Instrument development and validation by experts.
   - Pilot testing to check clarity of questionnaire and test items.
   - Ethical clearance and participant consent collection.
2. Data collection stage
   - The mathematical belief questionnaire was administered to all 30 participants. Students were categorized into belief levels using the following statistical grouping.

Table 2. Mathematical belief categories (Dündar, 2015)

Low	Middle	High
$S_1 = \bar{X} - \frac{s}{4}$	$S_1 < S_2 < S_3$	$S_3 = \bar{X} + \frac{s}{4}$

   - The conceptual understanding test was then given to assess students' understanding of limits at a point and at infinity, in both algebraic and graphical forms, using indicators from Table 1.
   - Six students (two from each belief group) were selected for semi-structured interviews. These interviews aimed to delve deeper into their reasoning, misconceptions, and conceptual understanding.
3. Data analysis stage
   - The belief questionnaire scores were analyzed using the formula from Table 2 to determine belief levels.
   - Test results were analyzed using a rubric aligned with conceptual indicators, focusing on students' use of concepts, methods, and explanations.
   - Interview transcripts were analyzed using the Huberman (2014) model, consisting of: Data reduction to selecting relevant information and coding key themes. Data display to organizing findings in matrices or descriptive profiles. Conclusion drawing and verification to interpreting patterns and verifying consistency across data sources.

Results

Based on the Mathematical Belief Questionnaire (MBQ) administered to 30 mathematics education students at Syiah Kuala University, students were categorised into three groups: High Mathematical Belief (HMB) with five students, Medium Mathematical Belief (MMB) with 16 students, and Low Mathematical Belief (LMB) with nine students. Eight participants were selected based on communication fluency: S1 and S2 from the HMB group, S3, S4, S5, and S8 from the MMB group, and S7 and S10 from the LMB group (Table 3).

Table 3. Subject group mathematical belief

Subject Group	Subject Code
High Mathematical Belief (HMB)	PW(S1), SP (S2)
Medium Mathematical Belief (MMB)	EV (S3), SG (S8), SD (S4),
Low Mathematics Belief (LMB)	NJ (S7), SF (S10)

Based on Table 3 grouping of beliefs and fluency of communication, the interview subjects were determined, namely students with the code PW as S1 and SP students as S2 from the HMB group, EV as S3, SD as S4, and S5, and the subjects of the LMB group were NJ as S7 and SF as S8.

The results of the Test of Problem on Limit Function (TPLF) showed that six students had a high understanding of limit of function (HULF), 15 had a medium understanding (MULF), and nine had a low understanding (LULF), as presented in Table 4.

Table 4. Percentage grouping understanding of limit of function

Understanding Group	Frequency (Percentage)
High Understanding of Limit of Function (HULF)	6 (20%)
Medium Understanding of Limit of Function (MULF)	15 (50%)
Low Understanding of Limit of Function (LULF)	9 (30%)

The results of the interviews conducted with the selected participants revealed several important findings related to their conceptual understanding of the limit of a function and their mathematical beliefs.

The following is an excerpt from the interview results between the researcher (P) and S2 as follows.

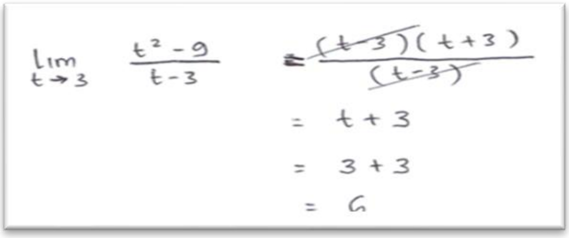

Figure 1. S2 Answer to problem 2

P01 : How do you get the limit value?
S201 : First, I factorized it into (t-3)(t+3), I cancel t-3, I got limit t+3
P02 : Why did you factorize it in??
S202 : If we substitute the value of t we will get zero per zero, so I factorized it first.
P03 : Why did you cancel t-3?
S203 : Because it was same?
P04 : Do you have other reason?
S204 : m...m.. because t is close to 3, it is not equal to 3.
P05 : then, wat did you do?
S205 : I substituted the values of x that are close to 3, the limit value is 6..
P06 : Do you often solve the similar problems??
S206 : Often sir, as soon as I saw a question like this I immediately did it.

The results of answering question 5 regarding and explaining the determination of values $\lim_{x \to 0} x + 1 + \frac{1}{10^{30}x}$ as Figure 2.

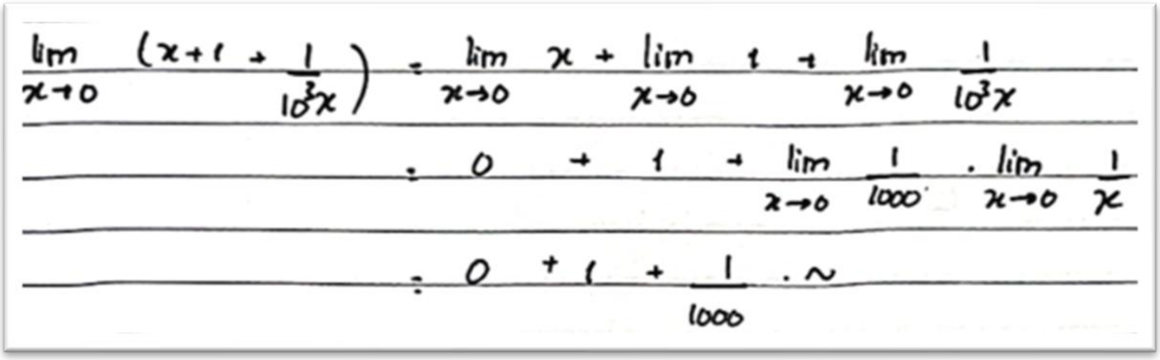

Figure 2: S2 Answer to question 5

P01 : Explain how to obtain the limit value?
S201 : This limit is first separated into limit x plus limit 1 plus limit 1 per 10 to the power of 10 x.
P02 : Why did you separate it?
S202 : to make it easier to find the limit
P03 : do you have any other reason?
S203 : No, Sir.
P04 : Where did you get 0,1 from?
S204 : limit of constant function 0 is equal to zero, limit one is equal to 1, limit 0 is equal to zero.
P05 : Why is the zero limit equal to zero?
S205: the property of a constant limit is the same as the value itself.

In solving question 6 about what is the value of $\lim_{x \to 2} f(x)$ if the graph is given is presented in Figure 3 below.

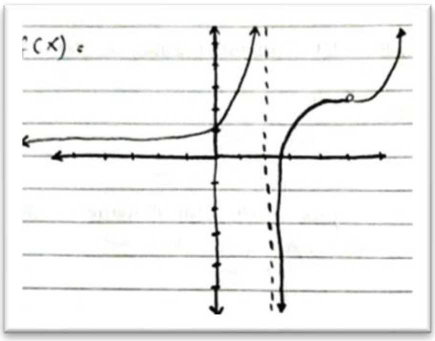

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Figure 3. The graph for question 6, 7, dan 8

The following is an excerpt from an interview between researcher (P) and S01 as follows:

P01 : Can you explain how to obtain the limit value?

S101 : As before, sir, I substituted the value of x that were close to 2 from the right, from the left, the graph [subject shows the graph in Figure 3] got a value around 2. The limit is 2.

P02 : why did you substitute the value of $x\ is\ 2$??

S102 : It is given in the question, Sir. x is approach to 2.

The results of completing S1 related to question 9, namely determining and explaining the value $\lim_{x\to 5} f(x)$ presented in Figure 4 below.

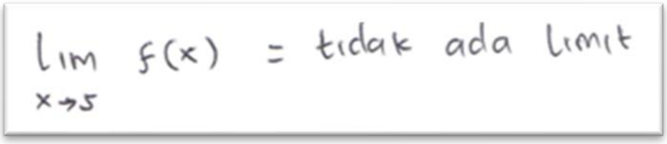

Figure 4: S1 Answer to Question 9

The following is an excerpt from an interview between researcher (P) and S1 as follows.

P01 : Can you explain how to obtain the limit value?

S101 : By substituting values of x close to 5, but this graph [shows the graph of Figure 3] is blank at the point x equals 5, the limit does not exist.

P02 : What does it mean there is none?

S102 : There is no limit because the empty graph is undefined.

Discussion

In general, based on the data in Table 4 on solving the function limit, HMB students are able to explain the determination of the function limit at one point, but students in the MMB and LMB categories generally make mistakes in researching the function limit at infinity. This is in accordance with Laja (2022), who found that mathematics education students made three mistakes related to trigonometric limit material, namely conceptual, operational, and principle errors. Based on the results of the function limit solution data and interview excerpts, 3 (three) groups of subject categories were obtained. Three levels of conceptual understanding of the limit of a function are discussed, namely conceptual understanding of the limit of a function based on HMB, MMB, and LMB. The discussion is as follows.

Students' conceptual understanding based on high mathematical beliefs

The conceptual understanding of the limit function among students with high mathematical belief (HMB) shows that they are able to determine the limit of a linear or polynomial function at a point by applying the substitution theorem. This method is chosen because they understand that polynomial functions are continuous, so substitution yields the correct limit value. In rational functions, they simplify expressions through factoring and cancel common terms, with the reasoning that t approaches r, but $t \neq r$. This reflects their understanding of function behavior near a point. Such reasoning illustrates a strong connection between procedures and concepts, which aligns with Maries et al. (2016), who found that students with high mathematical beliefs tend to integrate symbolic skills with conceptual understanding when solving limit and continuity problems.

The HMB subject has a high level of thinking assumption regarding the understanding of the limit theorem of a polynomial function and its logical use as a tool, a rule in solving problems. This is in accordance with the research of Ozturk and Guven (2016) that someone who has consistent logical beliefs is able to formulate concepts, ideas, and formulas as the right strategy in solving problems, while someone who has memorization beliefs and is able to remember procedures is able to solve similar problems that have been given before. Based on the solution in Figure 1 and the interview, it was obtained that the problem was solved using the factoring method, using the substitution method. Furthermore, the subject explained that the function limit is a real number L that is approached by the ordinate $\gamma(x)$ when x approaches the number r but $x \neq r$.

The Subject's understanding in solving question 3 is about what the value is $\lim_{x\to\sim} \frac{x^5}{(1{,}1)^x}$ zero is obtained by using Hopital's theorem on the grounds that the limit formed $\frac{\sim}{\sim}$. Then, the subject used the derivative until the value 0 is obtained. The subject has high ability and experience in working on limit of function at infinity. Figure 4 is one of the results of solving the subject's questions of $\lim_{x\to\sim} \frac{x^5}{(1{,}1)^x}$. This approach shows the subject's solid grasp of how to handle exponential growth in the denominator and polynomial growth in the numerator, a reasoning strategy consistent with Byerley and Thompson (2017), who noted that successful limit solvers often integrate symbolic manipulation with conceptual strategies when addressing infinity.

In completing question 5 regarding determination $\lim_{x\to 0} x + 1 + \frac{1}{10^3 x}$, The function limit was solved by subject S2 using the limit theorem from the addition of two or more functions, then the subject concluded $\lim_{x\to 0} x + 1 + \frac{1}{10^3 x}$ can be stated to be $\lim_{x\to r0} x + \lim_{x\to 0} 1 + \lim_{x\to 0} \frac{1}{10^3 x}$. They correctly applied the identity and constant limit theorems for the first two components. However, they misjudged the third component, $\lim_{x\to 0} \frac{1}{10^3 x}$, and concluded the overall limit to be 1, which signals a misunderstanding of divergent behavior. Nevertheless, their approach indicates an appropriate use of structural decomposition and theorem-based reasoning. This aligns with recent findings by Adhikari (2020), who observed that many students could correctly decompose limit expressions yet still misinterpret unbounded terms if they didn't fully grasp divergence, particularly those involving $\lim_{x\to 0} \frac{1}{x}$. These studies confirm the importance of combining procedural decomposition with deeper conceptual insight in advanced calculus learning.

Based on the completion of the question $\lim_{x\to 2} f(x)$ and the interview show that $\lim_{x\to 2} f(x)$ does not exist that determined by the subject by observing and analyzing the graph $f(x)$ away from line $x = 2$ indefinitely as x approaches 2 from the left, not with the graph of $f(x)$ away from the line $x = 2$ indefinitely as x approaches 2 from the right. So, the subject is able to explain the determination of the limit of a function by involving graphs and using the relationship between the concepts of left and right limits. In solving the question of how much is $\lim_{x\to 5} f(x)$ equal to 3, the subject explains by involving the graph of figure 1 because the value of the left limit is the same as the right limit, so it is concluded that $\lim_{x\to 5} f(x)$is equal to 3. This aligns with Sari (2017), who found that visual representations of functions help students accurately determine limit values at specific points. Similarly, Gebeyehu et al. (2021) emphasized that visualization is not merely a support tool but a core element in conceptual mathematical reasoning. These findings suggest that HMB students possess the ability to explain limits by integrating graphical analysis with conceptual knowledge

In solving the limit at infinity, the subject worked on Question 2, which asked for the value of $\lim_{x\to\sim} \sin x$. The subject correctly concluded that the limit is indefinite, based on the graphical behavior of the sine function that oscillates indefinitely and does not converge. This response demonstrates the subject's use of graphical reasoning in interpreting the behavior of trigonometric functions at infinity. The finding is consistent with Laja (2022) who emphasized the importance of mastering foundational concepts when studying trigonometric limits.

For Question 5, which involved evaluating $\lim_{x\to\sim} \frac{x^5}{(1,1)^x}$, the subject identified the expression as an indeterminate form $\frac{0}{0}$, applied L'Hôpital's Rule, and continued differentiating until the result approached zero. This solution shows the subject's ability to apply analytical strategies to limits involving exponential and polynomial expressions. It supports the findings of Putri et al. (2022) that the advantages of capable students can implement trial and error strategies. The trial strategy is based on ideas and logical thinking to form the understanding that each student has in each capable category.

In Question 7, the subject analyzed $\lim_{x\to\sim} f(x)$ using a graph (Figure 1). They correctly concluded that the limit tends to infinity, explaining that the graph increases without bound as x approaches infinity. The subject interpreted the notation "$x \to \sim$" as indicating unbounded growth, which reflects a conceptual understanding of infinite behavior. This observation is in line with A'zima et al. (2019) who found that students with strong procedural skills are able to apply methods appropriately in solving limit problems.

Additionally, the subject believes that calculus is a logical discipline, and that understanding it involves internalizing facts, concepts, and procedures, as well as being able to explain the relationships among them using appropriate principles and graphical tools. This belief reflects an epistemological view that mathematics is not merely about computation, but also about reasoning and structure. Such a view encourages deeper engagement with mathematical ideas and fosters greater flexibility in problem solving. This aligns with Usman et al. (2020), who reported that students who perceive calculus as a logical and structured field tend to demonstrate stronger conceptual reasoning, especially when solving non-routine problems that require more than procedural recall.

The subject's responses also indicate frequent experience working with limit theorems, which contributes to their ability to solve problems conceptually and explain the connections between ideas and solution steps. This experience is evident in the way the subject selects appropriate strategies, justifies the use of limit properties, and transitions between symbolic and graphical representations with confidence. Such fluency suggests that their understanding is not merely procedural, but grounded in a coherent mental framework of how limits behave under various conditions. This is consistent with the findings of Sebsibe and Feza (2020), who highlighted that most students develop conceptual knowledge of limits through repeated exposure, reflective thinking, and continuous engagement in solving a variety of problems. These practices help students internalize fundamental concepts, recognize patterns across different limit scenarios, and strengthen their mathematical intuition over time.

Furthermore, the subject demonstrates logically consistent thinking, which has been shown to correlate with higher-level problem-solving ability compared to students who rely only on memorization or procedural steps. This is supported by Soesanto et al. (2021) and further reinforced by Arnal-Palacián (2022), who noted that the concept of limits at infinity can be validated through two-way (one-way and reverse) approaches when students are able to construct and justify functional behavior. Finally, the subject’s emphasis on logical relationships in mathematics echoes the view of Cifarelli et al. (2010), who argued that beliefs grounded in logical reasoning enhance students' ability to solve complex problems conceptually and independently.

Conseptual understanding of limit function of studrnts with medium mathematical belief (MMB)

The conceptual understanding of limits among students with Medium Mathematical Belief (MMB) is demonstrated primarily through their procedural approaches. The subject was able to determine the limit of a linear function at a specific point using the substitution theorem, justified by the understanding that a linear function $\gamma(x)$ is continuous, and thus, direct substitution yields the limit value L. This procedural fluency, however, often lacks deeper theoretical justification, as described by Prendergast et al. (2018), who noted that students with moderate mathematical beliefs frequently rely on procedures without fully understanding their conceptual underpinnings.

When solving question 2, regarding the limit of a rational function, the MMB subject simplified the function through factoring and cancellation of identical factors in the numerator and denominator. Although they correctly used the substitution method afterward, the subject described the resulting $\lim_{x \to 3} f(x) = 6$ as an approximate value, indicating an incomplete conceptual understanding. This aligns with research by Kristanto et al. (2019) who observed that students with moderate mathematical beliefs may execute algebraic steps correctly but still lack a full conceptual grasp of the limit's meaning.

In addressing question 5, $\lim_{x \to 0} x + 1 + \frac{1}{10^3 x}$, the subject appropriately decomposed the limit into separate terms: $\lim_{x \to 0} x$, $\lim_{x \to 0} 1$, dan $\lim_{x \to 0} \frac{1}{10^3 x}$. However, the student was unable to justify why this decomposition is valid or explain the relationships among these terms using relevant limit theorems. They proceeded to substitute directly, mistakenly concluding the entire limit to be equal to 1 without recognizing the divergent nature of $\frac{1}{10^3 x}$. This aligns with findings from Sebsibe et al. (2019), who observed that many calculus students exhibit the misconception that “limit is a substitution,” often leading to procedural overgeneralization and failure to interpret divergence correctly.

The subject admitted that their calculus learning relies heavily on provided examples and formulas rather than logical reasoning. This reflects findings by Prendergast et al. (2018), who reported that students with moderate mathematical beliefs tend to see calculus as formula-driven rather than logically structured. Additionally, the subject showed some attempts at using logical methods and strategies in problem-solving, aligning with observations by Putri et al. (2022), who noted that moderately capable students often rely on trial-and-error strategies rooted in logical yet incomplete conceptual reasoning.

In addressing Question 6, the subject was asked to evaluate $\lim_{x\to 2} f(x)$ using the graph provided in Figure 1. The subject correctly identified that the limit does not exist, explaining that the left-hand and right-hand limits differ. However, their reasoning was presented separately and without explicitly referencing the graphical evidence provided, particularly the behavior of the graph as it diverges indefinitely from the line $x = 2$. This indicates a partial gap in integrating graphical interpretation with conceptual reasoning, consistent with the observation by Afgani et al. (2017) that students often struggle to effectively articulate graphical behavior when limits involve divergence or discontinuity.

In solving question 8 about what is the value of $\lim_{x\to 5} f(x)$, the limit value is obtained by the subject by involving the graph in figure 1 because the left limit value is the same as the right limit, so the subject concludes that $\lim_{x\to 5} f(x)$ is equal to 3. Thus, the conceptual understanding of the MMB subject in solving limits is not yet complete. Recent research indicates that while visual aids can bolster limit comprehension, students with moderate mathematical belief often show inconsistent use of graphical information. Wakhata et al. (2023) demonstrated that when students engage with multiple representations, including symbolic, verbal, and graphical modes, their conceptual understanding improves, particularly for limits involving both continuity and discontinuity. This underscores the need for explicit instructional support that strengthens students’ ability to connect graph behavior with formal limit concepts.

Based on the solution and interview results for Question 2, which asked about the limit at infinity $\lim_{x\to\sim} \sin x$, the MMB subject demonstrated a conceptual misunderstanding. The subject incorrectly concluded that the limit is "indefinite" and attempted to justify it using substitution, reasoning that the value of the sine function becomes larger as x increases. However, this reasoning is flawed, as $\sin x$ is a bounded, oscillatory function that does not approach a single value as $x \to \sim$. The subject also failed to relate this to the graphical behavior of the function, indicating limited conceptual grounding. This aligns with the findings of (Jones, 2015), who noted that many students struggle to interpret oscillatory limits, especially when they overly rely on symbolic manipulation without connecting it to function behavior.

In solving question 5 regarding $\lim_{x \to \sim} \frac{x^5}{(1,1)^x}$ is equal to zero, the MMB subject understands by applying the function limit quotient theorem, then intuitively the graph of x^5 when x increases without limit and the graph of$(1,1)^x$ when x increases without limit. The meaning of "$x \to \sim$" understood by the MMB subject to increase without limit. The MMB subject made a mistake in explaining how to obtain the limit value because the subject rarely found examples of solving such problems in calculus learning. According to Motseki and Luneta (2024), students in similar contexts often exhibit a superficial understanding of growth comparisons, frequently mixing procedural fluency with incomplete conceptual insight.

Conseptual undersatnding of limit of functions of students with low mathematicsal beliefs (LMB)

The conceptual understanding of the limit of the function by the LMB (Low Mathematical Belief) subject is reflected in their ability to determine the limit of a linear function at a specific point when given in the form of an equation. The subject applied the substitution theorem, reasoning that since the function $\gamma(x)$ is linear, direct substitution yields the limit value as a real number L. This indicates a procedural but limited conceptual grasp of the limit concept. As noted by Prendergast et al. (2018), students with weaker mathematical beliefs tend to rely heavily on memorized rules and symbolic manipulation without fully understanding the underlying concepts.

In solving question 2 about determining the limit of a rational function, the limit of the function is determined by the LMB subject by simplifying the rational function using the factoring method, then deleting the same factors in the numerator and denominator on the grounds that the factors are the same, then the limit value is obtained as the number L by substitution. The value $6 = \lim_{x \to 3} f(x)$ is understood by the LMB subject as an approximation. This behavior is consistent with findings by Kristanto et al. (2019), who reported that students with moderate to low mathematical beliefs may perform procedural steps correctly but still lack deep comprehension of the concepts, often interpreting exact limits as estimations.

Furthermore, the solution to question 5 about what is the value of $\lim_{x \to 0} x + 1 + \frac{1}{10^3 x}$, the limit of the function is solved by the LMB subject by stating $\lim_{x \to 0} x + 1 + \frac{1}{10^3 x}$ become $\lim_{x \to r0} x + \lim_{x \to 0} 1 + \lim_{x \to 0} \frac{1}{10^3 x}$ that equal to 1. However, the subject was unable to explain the reason for the change in the form of the limit. Furthermore, by using the substitution method on each $\lim_{x \to 0} x$ and $\lim_{x \to 0} 1$ is obtained that $0 + 1 + 0$ is equal to 1, but the subject is unable to explain its relation to the limit theorem, then the limit value of $\lim_{x \to 0} \frac{1}{x}$ is obtained equal to zero defined by the subject's reason that the substitution x is equal to 0. This type of error is commonly observed among students with lower mathematical beliefs, who often rely on memorized procedures without a full grasp of the underlying principles Putri et al. (2022). Similarly, research by Tanjung et al. (2025) highlights that students often misinterpret limits involving undefined or infinite behavior when they lack sufficient conceptual grounding.

The LMB subject made a mistake in calculating the limit value of an infinite rational function. The subject admitted that the lack of examples given during learning contributed to their difficulty in understanding and solving such problems. This suggests that procedural knowledge alone, without sufficient conceptual grounding and guided practice, is insufficient for mastering abstract concepts like limits at infinity. The subject's struggle reflects a broader issue observed in students with low mathematical beliefs, who often rely on memorization rather than developing a deep conceptual understanding. This finding is in line with the study of Wewe (2020), which emphasizes that many students experience significant difficulties in grasping core calculus concepts, particularly the notion of limits of functions.

The LMB subjects' understanding in solving limit at infinity, question 2, about what is the value of $\lim_{x \to \sim} sinx$, is determined by using the method of substituting the value of x into the sine function equal to 0 on the grounds of learning experience in solving limit problems, but the subject is unable to explain the meaning of the concept of trigonometric limits at infinity. In addition, the LMB subject has a wrong understanding in determining the value of the infinite limit. Furthermore, the understanding of the LMB subject in solving question 5, $\lim_{x \to \sim} \frac{x^5}{(1,1)^x}$. The LMB subject made a mistake in determining the value of $\lim_{x \to \sim} \frac{x^5}{(1,1)^x}$ that the error in using theorems, concepts, and being unable to explain the meaning of the concept of high limits and their relationship. Based on the analysis, it was obtained that the subject of low mathematical confidence did not have a conceptual understanding in solving trigonometric limit problems and polynomial functions. Tamba et al. (2022) stated that the low conceptual understanding of calculus of prospective mathematics teachers provides important implications for learning approaches that encourage the development of conceptual understanding of mathematics.

Subjects with low mathematical beliefs do not have a conceptual understanding in solving limit of function at one point and at infinity. They have low logical and consistent beliefs in solving function limit problems at one point and at infinity in everyday learning. In fact, the subject's confession during the LMB subject interview rarely solves function limit problems if they are known in graphical form, as a result, the LMB subject if the function graph is known. This is in accordance with the findings of Soesanto et al. (2021) that students with low logical mathematical beliefs have lower problem-solving abilities than memorization and procedural beliefs. In addition, this is also in accordance with the findings of Laja (2022) that mathematics education students make 3 (three) mistakes related to trigonometric limit material, namely conceptual, operational, and principle errors. This is in accordance with the findings of Szydlik (2000) that students with medium mathematical beliefs think that calculus does not always work logically and systematically.

Conclusion

This study revealed three distinct profiles of students' conceptual understanding of the limit of functions, categorised by their levels of mathematical belief: high, medium, and low. Students with high mathematical beliefs (HMB) demonstrated a comprehensive and integrated understanding of mathematics. They were able to determine limits using algebraic, graphical, and theoretical approaches and could explain the relationships between concepts and solution procedures. These students demonstrated strong logical reasoning, consistently applying definitions, theorems, and principles to solve limit problems, including those involving infinity. They were also capable of generalising their results, reflecting deep and abstract understanding. In contrast, students with moderate mathematical beliefs (MMB) were able to solve limit problems both algebraically and graphically but struggled to explain the conceptual underpinnings of their solutions. Their understanding remained partially procedural and often depended on previously encountered examples. Similarly, students with low mathematical beliefs (LMB) rely heavily on procedural steps without demonstrating a strong grasp of the underlying concepts. They faced difficulties, especially in interpreting limits at infinity, and were unable to establish meaningful connections between methods and mathematical reasoning.

These findings indicate that students' conceptual understanding of calculus is strongly influenced by their mathematical beliefs. As students develop more logical and confident beliefs, their ability to reason, explain, and generalise has been shown to improve. The novelty of this study lies in its integrative approach, which connects belief systems with both conceptual and procedural aspects of understanding limits. Unlike prior research that isolates errors or focuses narrowly on formal knowledge, this study provides a comprehensive view of students' thinking across different belief levels. Accordingly, calculus instruction should go beyond procedural fluency by fostering students' capacity to explain their reasoning and reflect on the relationships among concepts, procedures, and solutions. Additionally, although self-reported data and qualitative interviews offer deep and nuanced insights into students' thought processes, they are inherently subjective and may be influenced by students' ability to articulate their understanding clearly, potentially limiting the accuracy of findings.. Despite these limitations, this study offers valuable insights into the role of beliefs in conceptual learning and supports the integration of belief-oriented strategies in calculus instruction. Future research should explore larger and more diverse samples and evaluate the impact of targeted interventions.

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Conflicts of Interest

The authors declare no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies, have been completed by the authors.

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Author Contributions

Usman: Conceptualization, writing - original draft, editing, and visualization; **Aiyub:** Writing - review & editing, formal analysis, and methodology; **M. Hasbi:** Validation and supervision.

References

Adhikari, K. P. (2020). Difficulties and misconceptions of students in learning limit. *Interdisciplinary Research in Education*, *5*(1&2), 15–26. https://doi.org/10.3126/ire.v5i1&2.34731

Afgani, M. W., Suryadi, D., & Dahlan, J. A. (2017). Analysis of undergraduate students' mathematical understanding ability of the limit of function based on apos theory perspective. *Journal of Physics: Conference Series*, *895*(1), 012056. https://doi.org/10.1088/1742-6596/895/1/012056

Angel-Cuervo, Z. M., Briceño-Martínez, J. J., & Bernal-Ballén, A. (2024). Validation of a questionnaire to evaluate mathematics teachers' beliefs about mathematics, teaching, and learning. *Eurasia Journal of Mathematics, Science and Technology Education*, *20*(5), 1–15. https://doi.org/10.29333/ejmste/14447

Arnal-Palacián, M. (2022). Infinite limit of a function at infinity and its phenomenology. *Annales Universitates Paedagogicae Cracoviensis. Studia Ad Didacticam Mathematicae Pertinentia*, *14*, 25–41. https://doi.org/10.24917/20809751.14.3

Arsyad, N., Ramadhana, Y., & Assagaf, S. F. (2021). Students' factual understanding on the concept of limit. *Journal of Physics: Conference Series*, *1899*(1), 012148. https://doi.org/10.1088/1742-6596/1899/1/012148

A'zima, R., Roza, Y., & Maimunah. (2019). *Analisis kemampuan pemahaman konsep matematis siswa dalam menyelesaikan soal limit fungsi* [Analyzing students' conceptual understanding in solving problems on function limits]. *7*(2), 12–18.

Beynon, K. A., & Zollman, A. (2015). Lacking a formal concept of limit: advanced non-mathematics students' personal concept definitions. *The Research Council on Mathematics Learning*, *8*(1), 47–62.

Busetto, L., Wick, W., & Gumbinger, C. (2020). How to use and assess qualitative research methods. *Neurological Research and Practice*, *2*(14), 1–10. https://doi.org/10.1186/s42466-020-00059-z

Byerley, C., & Thompson, P. W. (2017). Secondary mathematics teachers' meanings for measure, slope, and rate of change. *Journal of Mathematical Behavior*, *48*, 168–193. https://doi.org/10.1016/j.jmathb.2017.09.003

Cifarelli, V., Goodson-espy, T., & Chae, J. (2010). Associations of students' beliefs with self-regulated problem solving in college algebra. *Journal of Advanced Academics*, *21*(2), 204–232.

Dündar, S. (2015). Mathematics teacher- candidates' performance in solving problems with different representation styles: the trigonometry example. *Eurasia Journal of Mathematics, Science and Technology Education*, *11*(6), 1379–1397. https://doi.org/10.12973/eurasia.2015.1396a

Faizah, H. (2019). *Profil pemahaman konseptual mahasiswa calon guru matematika tentang grup ditinjau dari kemampuan matematika* [Prospective mathematics teachers' conceptual understanding of groups based on their mathematical ability]. *4185*, 84–97.

Feudel, F., & Biehler, R. (2022). *Students ' understanding of the economic interpretation of the derivative in the context of marginal cost*. 437–468.

Gebeyehu, M., Atnafu, M., & Ejigu, T. (2021). Implementing geogebra integrated with multi-teaching approaches guided by the apos theory to enhance students ' conceptual understanding of limit in ethiopian universities. *Heliyon*, *7*(5), e07012. https://doi.org/10.1016/j.heliyon.2021.e07012

Huberman. (2014). *Qualitative data analysis* (H. Salmon, Ed.; 3rd ed.). SAGE Publication.

Jones, S. R. (2015). Calculus limits involving infinity: the role of students' informal dynamic reasoning. *International Journal of Mathematical Education in Science and Technology*, *46*(1), 105–126. https://doi.org/10.1080/0020739X.2014.941427

Klau, K. Y., Siahaan, M. M. L., & Simarmata, J. E. (2020). An identification of conceptual and procedural understanding: study on preservice secondary mathematics teacher. *Al-Jabar : Jurnal Pendidikan Matematika*, *11*(2), 339–350. https://doi.org/10.24042/ajpm.v11i2.7310

Kristanto, Y. D., Melissa, M. M., & Panuluh, A. H. (2019). Discovering the formal definition of limit through exploration in dynamic geometry environments. *Journal of Physics: Conference Series*, *1180*(1), 012004. https://doi.org/10.1088/1742-6596/1180/1/012004

Laja, Y. P. L. (2022). *Analisis kesulitan mahasiswa pendidikan matematika dalam menyelesaikan soal limit trigonometri* [Analyzing the difficulties of mathematics education students in solving trigonometric limit problems]. *11*, 37–48.

Maries, A., Lin, S.-Y., & Singh, C. (2016). *The impact of students' epistemological framing on a task requiring representational consistency*. 212–215. https://doi.org/10.1119/perc.2016.pr.048

Motseki, P., & Luneta, K. (2024). An error analysis of TVET students' responses to optimisation problems. *African Journal of Research in Mathematics, Science and Technology Education*, *28*(1), 134–152. https://doi.org/10.1080/18117295.2024.2341358

Muhtarom, M., Juniati, D., Siswono, T. Y. E., & Rahmatika, I. (2018). Teachers' and students' beliefs in mathematics at State Senior High School 5 Semarang. *Jurnal Riset Pendidikan Matematika*, *5*(1), 64–72. https://doi.org/10.21831/jrpm.v5i1.18734

NCTM. (2020). Standards for the Preparation of Secondary Mathematics Teachers. *The National Council of Teachers of Mathematics, Inc.*, *May*, 1–84.

Nurdin, Assagaf, & Arwadi. (2019). *Students ' understanding on formal definition of limit students ' understanding on formal definition of limit*. https://doi.org/10.1088/1742-6596/1752/1/012082

Oktaviyanthi, R., Herman, T., & Dahlan, J. A. (2018). *How does pre-service mathematics teacher prove the limit of a function by formal definition ? 9*(2), 195–212.

Ozturk, T., & Guven, B. (2016). Evaluating students' beliefs in problem solving process: A case study. *Eurasia Journal of Mathematics, Science and Technology Education*, *12*(3), 411–429. https://doi.org/10.12973/eurasia.2016.1208a

Prendergast, M., Breen, C., Bray, A., Faulkner, F., Carroll, B., Quinn, D., Carr, M. (2018). Investigating secondary students beliefs about mathematical problem-solving. *International Journal of Mathematical Education in Science and Technology*, *January*. https://doi.org/10.1080/0020739X.2018.1440325

Putri, D., Usodo, B., & Subanti, S. (2022). *Students' mathematical conceptual understanding : What happens to proficient students ? 2017*(November).

Saleh. (2018). *Pemahaman mahasiswa pada konsep limit fungsi berdasarkan kemampuan matematika* [Understanding of function limit concepts among students based on their mathematical ability]. *1*(x), 191–196.

Sari, P. (2017). Geogebra as a means for understanding limit concepts. *Southeast Asian Mathematics Education Journal*, *7*(2), 71–84. https://doi.org/10.46517/seamej.v7i2.55

Sebsibe, A. S., Bekana, T., & Dorra, T. (2019). Effect of error analysis and concept change strategy on students understanding of calculus concepts. *IEEE-SEM*, *7*(7), 369–385. https://www.researchgate.net/publication/349073918

Sebsibe, A. S., & Feza, N. N. (2020). Assessment of students' conceptual knowledge in limit of functions. *International Electronic Journal of Mathematics Education*, *15*(2), 1–15. https://doi.org/10.29333/iejme/6294

Soesanto, R. H., Putri, K., & Dirgantoro, S. (2021). *Kemampuan pemecahan masalah mahasiswa pada kalkulus integral dilihat dari keyakinan dan pengetahuan awal* *matematis* [Students' problem-solving ability in integral calculus viewed from mathematical beliefs and prior knowledge]. *7*(1), 117–129. https://doi.org/10.29408/jel.v7i1.2899

Sulastri, R., Suryadi, D., Prabawanto, S., Cahya, E., & Fitriani, F. (2022). Zone of concept image differences in infinite limits at undergraduate level. *Jurnal Didaktik Matematika*, *9*(1), 1–21. https://doi.org/10.24815/jdm.v9i1.24709

Szydlik, J. E. (2000). *Mathematical beliefs and conceptual understanding of the limit of a function*. *31*(3), 258–276.

Tamba, K. P., Appulembang, O. D., & Listiani, T. (2022). *Korelasi antara keyakinan belajar dan pemahaman konseptual kalkulus pada calon guru matematika* [The correlation between learning beliefs and conceptual understanding of calculus in prospective mathematics teachers]. *JNPM (Jurnal Nasional Pendidikan Matematika)*, *6*(1), 20. https://doi.org/10.33603/jnpm.v6i1.5315

Tanjung, J. Y., Ade, R., Simanjuntak, P., Manullang, J. C., Turnip, L., & Simanullang, M. C. (2025). *Analisis kesalahan mahasiswa dalam menyelesaikan soal limit fungsi berdasarkan teori kastolan* [Analyzing mathematics students' errors in solving function limit problems using kastolan's theory]. *Mandalika Mathematics and Education Journal*, *5*, 391. https://doi.org/10.29303/jm.v3i1.9014

Usman, Bambang, R. M., Hasbi, S. M., & Mardhiah, M. Z. (2020). Analysis of calculus learning beliefs and students' understanding about limit of functions: a case study. *Journal of Physics: Conference Series*, *1462*(1), 012024. https://doi.org/10.1088/1742-6596/1462/1/012024

Usman, Juniati, D., & Siswono, T. Y. E. (2017). Exploration the conception of prospective students teacher about limit of function. *AIP Conference Proceedings*, *1868*(August). https://doi.org/10.1063/1.4995162

Wabula, J. D., & Cahyono, E. (2017). *Analisis pemahaman konsep limit fungsi pada siswa sma berdasarkan jenis kelamin* [Analyzing high school students' conceptual understanding of function limits in relation to gender differences]. *Seminar.Uny.Ac.Id*, 191–196.

Wakhata, R., Balimuttajjo, S., & Mutarutinya, V. (2023). Enhancing the learning of limits of functions using multiple representations. *Mathematics Teaching Research Journal*, *5*(2), 182–202.

Wewe, M. (2020). The profile of students' learning difficulties in concepts mastery in calculus course. *Desimal: Jurnal Matematika*, *3*(2), 161–168. https://doi.org/10.24042/djm.v3i2.6421

Winarso, W., & Toheri, T. (2017). A case study of misconceptions students in the learning of mathematics: The concept limit function in high school. *4*(1), 120–127.

Yoon, B., & Uliassi, C. (2022). "researcher-as-instrument" in qualitative research: The complexities of the educational researcher's identities. *Qualitative Report*, *27*(4), 1088–1102. https://doi.org/10.46743/2160-3715/2022.5074