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Developing a PMRI-based augmented reality learning trajectory to support students' problem-solving skills in circle geometry

Trisna Wulandari, Ely Susanti *, Cecil Hiltrimartin

Mathematics Education Study Program, Sriwijaya University, South Sumatra, Indonesia

* Correspondence: ely_susanti@fkip.unsri.ac.id

Abstract

This study reports a validation study within design research aimed at developing a PMRI-based augmented reality (AR) learning trajectory to support high school students' problem-solving skills in circle geometry. The subjects of this study were 35 eleventh-grade students at a senior high school in Palembang. The research employed a design research methodology, specifically a validation study, consisting of three phases: preliminary design, design experiment, and retrospective analysis. Data were collected from worksheets, classroom observations, and problem-solving tests, and analyzed descriptively using qualitative and quantitative approaches. This study produced a learning trajectory consisting of a sequence of contextual and AR-assisted learning activities that guided students from exploring real-world circular situations to formal circle concepts. The implementation of this trajectory resulted in measurable improvements in students' problem-solving skills, as indicated by an increase in mean test scores from 19.37 in the pre-test to 66.51 in the post-test. Students demonstrated strong performance in understanding problems, devising solution plans, and carrying out solution strategies, with achievement percentages ranging from 71% to 100% on these indicators. However, students' evaluation and reflection skills remained relatively low, indicating the need for greater emphasis on reflective activities within the learning trajectory.

Keywords: augmented reality; circles geometry; learning trajectory; PMRI; problem-solving

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Introduction

Problem-solving ability is a fundamental competence in mathematics education (Klang et al., 2021; Utami et al., 2024), as it enables students to analyze situations, develop strategies, and apply mathematical concepts to real-world problems (Choirunisa & Susanti, 2024; Hartmann et al., 2021; Kohen & Nitzan-Tamar, 2022). In the context of 21st-century education, problem solving is closely associated with critical and creative thinking skills that are essential for addressing complex and non-routine tasks (González-pérez & Ramírez-montoya, 2022; Nilimaa, 2023; Susanti et al., 2025). International mathematics education standards, such as those proposed by the National Council of Teachers of Mathematics (NCTM), emphasize problem solving as a core process in mathematics learning (Malangtupthong et al., 2022). However, empirical studies indicate that many students still experience difficulties in solving contextual and non-routine mathematical problems, highlighting the need for instructional approaches that explicitly support the development of students' problem-solving skills (Baiduri et al., 2020; Wijayanti et al., 2025; Yayuk et al., 2020).

However, previous studies indicate that Indonesian students' mathematical problem-solving skills remain relatively low (Wijayanti et al., 2025; Yayuk et al., 2020). Many students are not accustomed to solving non-routine and contextual problems, resulting in difficulties in understanding problems, planning solution strategies, and explaining their reasoning processes (Yayuk et al., 2020). Empirical findings show that 56.25% of students make errors in understanding contextual problems, while 62.50% experience difficulties in applying problem-solving strategies (Sriwahyuni & Maryati, 2022). These challenges are influenced by limited prior knowledge and low learning engagement (Sakir & Kim, 2020), as well as instructional practices that remain predominantly teacher centered and lack variation (Tsegaw et al., 2021). Consequently, students' problem-solving competencies are not optimally developed (Hartmann et al., 2021), and opportunities to engage in meaningful contextual problem-solving activities in classrooms remain limited (Kohen & Nitzan-Tamar, 2022).

One mathematical topic that strongly requires problem-solving skills is circle geometry, particularly the topic of tangents (Arin et al., 2024; Juniarti et al., 2022; Lestari et al., 2021). The circle–tangent subtopic is an essential part of the curriculum, serving as a foundation for advanced mathematical concepts such as analytic geometry and trigonometry, and it has important real-world applications in areas such as vehicle trajectory design, navigation, and architecture (Kugblenu, 2022). Despite its importance, students frequently experience difficulties in learning circle material, including challenges in understanding problems, identifying relevant elements, planning strategies, and avoiding conceptual errors (Arin et al., 2024; Ostian et al., 2023). Limited conceptual understanding further hinders students' ability to apply circle concepts to spatial and geospatial problems, while the lack of interactive and contextual learning media remains a significant barrier (Yuan et al., 2023).

From a learning theory perspective, the teaching of circle geometry requires instructional approaches that support students in constructing concepts progressively from concrete experiences toward formal mathematical representations. Indonesian Realistic Mathematics Education (PMRI), which is adapted from Realistic Mathematics Education (RME), emphasizes the use of meaningful real-life contexts, students' active participation, and guided reinvention of mathematical ideas (Firsta & Susanti, 2024; Komariyatiningsih et al., 2023; Zulkardi et al., 2020). PMRI has been shown to support students' conceptual understanding and problem-solving abilities by connecting abstract mathematical concepts with situations that are familiar to students (Komariyatiningsih et al., 2023). In circle learning, PMRI facilitates students' understanding of geometric relationships through contextual modeling and gradual formalization processes (Arin et al., 2024).

The effectiveness of PMRI can be strengthened through the integration of 21st-century digital technology, particularly Augmented Reality (AR) (Aditya et al., 2022). AR provides interactive three dimensional visualizations that allow students to observe and manipulate geometric objects dynamically, thereby supporting deeper conceptual understanding (Malalina & Kesumawati, 2013; Nisa et al., 2019). Previous studies have demonstrated that PMRI-based learning supported by AR can enhance students' motivation, engagement, and problem-solving skills (Andzin et al., 2024; Meryansumayeka et al., 2022). AR is especially suitable for geometry learning because it enables students to visualize circles and tangents as concrete spatial objects, helping them reduce cognitive load and improve representational understanding (Li et al., 2022; Nisa et al., 2019). Empirical evidence also indicates that AR-assisted mathematics learning can significantly improve students' problem-solving performance (Nindiasari et al., 2024). Beyond visualization, AR supports problem-solving by enabling students to explore, test, and revise solution strategies dynamically.

Learning trajectories play a crucial role in designing coherent learning sequences that guide students from informal contextual understanding toward formal mathematical concepts (Gravemeijer & Cobb, 2006a). Learning trajectories using PMRI have been shown to increase students' enthusiasm for learning mathematics (Domu & Mangelep, 2020; Meryansumayeka et al., 2022). Previous research has shown that PMRI-based learning trajectories can facilitate gradual concept construction and improve students' problem-solving skills (Arin et al., 2024; Gee et al., 2018). Other studies have integrated AR into learning trajectories for various mathematical topics and reported positive impacts on students' conceptual understanding and engagement (Aditya et al., 2022; Andzin et al., 2024). The use of AR in mathematics education enriches the learning experience by increasing student engagement and conceptual understanding (Aditya et al., 2022), particularly in visualizing complex concepts such as spatial geometry and function graphs (Fokuo et al., 2023). However, research specifically focusing on the development of circle learning trajectories that integrate PMRI and AR to support high school students' problem-solving skills remains limited. Existing trajectory studies tend to emphasize conceptual understanding while providing limited analysis of problem-solving indicators, particularly in the evaluation and reflection stages. Moreover, although AR-based trajectory designs have been explored in other mathematical topics, their application to circle geometry despite its strong visual and spatial characteristics has not been sufficiently investigated.

Unlike previous studies that focus solely on PMRI-based instruction or augmented reality applications in geometry learning, this study integrates both approaches within a validated learning trajectory framework. Therefore, this study aims to develop a PMRI-based learning trajectory supported by Augmented Reality on circle material that can support Grade XI senior high school students' problem-solving skills. Based on this purpose, the research questions of this study are as follows: (1) How is a PMRI-based learning trajectory supported by Augmented Reality on circle material designed to support the problem-solving skills of Grade XI senior high school students? (2) What is the role of a PMRI-based learning trajectory supported by Augmented Reality on circle material in supporting the problem-solving skills of Grade XI senior high school students?

Methods

This research is a qualitative and quantitative research with method design research type validation studies that aims to produce a learning trajectory for class XI circle material using the Indonesian Realistic Mathematics Education (PMRI) approach assisted by Augmented Reality (AR) to support problem-solving abilities. This study employed a validation study as part of design research, aiming to examine the practicality and potential effectiveness of the designed learning trajectory. The validation phase focused on analyzing how the learning activities functioned in real classroom settings and how students engaged with the PMRI-based AR tasks. Qualitative data were used to analyze students' learning processes, while quantitative data were used to examine changes in students' problem-solving performance. Design research was chosen because it allows researchers to systematically design, test, and refine instructional interventions while closely examining students' learning processes (Gravemeijer & Cobb, 2006). The validation focus emphasizes the refinement of the designed learning trajectory through iterative cycles of implementation and analysis.

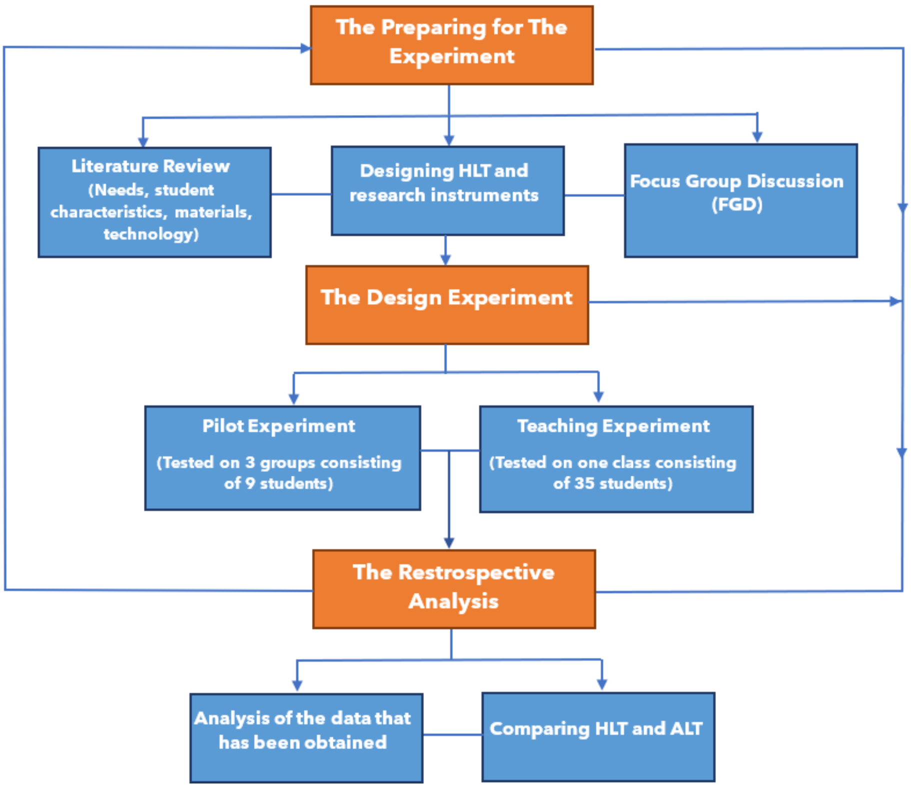


Figure 1. Design research stages

Based on Figure 1, the research design involved three stages (1) preparing for the experiment, (2) the design of the experiment, and (3) retrospective analysis (Gravemeijer & Cobb, 2006a). Figure 1 illustrates the overall stages of the design research, including the development of the Hypothetical Learning Trajectory (HLT), its testing through pilot and teaching experiments, and the comparison between the HLT and the Actual Learning Trajectory (ALT).

Research subjects

The research subjects consisted of 35 Grade XI students from SMA Srijaya Negara Palembang. The students were selected using purposive sampling, as they had already studied prerequisite topics such as basic geometry and algebra, which are essential for understanding circle and tangent concepts. The participants represented a heterogeneous group in terms of mathematical ability, based on prior classroom assessments conducted by the mathematics teacher. The students had limited prior experience with Augmented Reality in mathematics learning, making this context suitable for examining how AR supported PMRI learning influences problem solving skills.

Research procedures

The research procedures followed the stages of design research:

1. Preparing for the experiment
   This stage involved conducting a literature review on circle material, particularly tangents, PMRI principles, problem-solving skills, and the use of AR in mathematics learning. Based on this review, the researcher designed Student Activity Sheets (LAS) and developed a Hypothetical Learning Trajectory (HLT) consisting of learning objectives, learning activities, and conjectures about students' thinking.
2. Design experiment
   The design experiment consisted of two phases:
   a. Pilot Experiment, conducted with 9 students (three groups), aimed at testing and revising the initial HLT and learning materials. The pilot experiment was used to identify students' difficulties, refine task instructions, adjust mathematical representations, and revise the Hypothetical Learning Trajectory (HLT) before the teaching experiment.
   b. Teaching Experiment, implemented in one Grade XI class consisting of 35 students, aimed at examining how the revised HLT functioned in an actual classroom setting.
3. Retrospective analysis
   In this stage, the researcher analyzed all collected data by comparing the HLT with the Actual Learning Trajectory (ALT) that emerged during classroom implementation to identify consistencies, discrepancies, and necessary refinements.

To ensure validity and trustworthiness, data triangulation was applied by comparing student worksheets, classroom observations, interviews, and problem-solving test results. Peer discussions with mathematics education experts were conducted to review the learning trajectory design and interpretation of findings. In addition, iterative revisions through pilot and teaching experiments enhanced the credibility of the research results.

Data collection

Data were collected using multiple sources to comprehensively assess students' problem-solving skills:

1. Student activity sheets
   The worksheets contained contextual problems related to bicycle chain gears and circle tangents, guiding students through problem-solving stages such as understanding the problem, modeling the situation, planning strategies, implementing solutions, and drawing conclusions.
2. Tests
   The tests consisted of six open-ended questions on circle tangents designed to measure students' problem-solving abilities based on Polya's indicators: understanding the problem, devising a plan, carrying out the plan, and evaluating the solution.
3. Classroom observations
   Observations were conducted during the pilot and teaching experiments to capture students' learning processes, interactions, and use of AR. An observation protocol was used focusing on students' engagement, strategy development, collaboration, and difficulties encountered during problem-solving.
4. Interviews
   Semi-structured interviews were conducted with selected students to clarify their written responses, explore their reasoning processes, and gain deeper insight into their problem-solving strategies and reflections. Interview participants were selected based on students' problem-solving performance (high, medium, and low) and their level of engagement during the learning activities.

Data analysis

Data were analyzed using qualitative descriptive analysis. Student worksheets were examined to describe students' problem-solving processes at each stage of the learning trajectory. Classroom observations and interviews were used to clarify students' learning behaviors and reasoning processes during PMRI-based AR activities. Problem-solving test results were analyzed descriptively by calculating the percentage of students achieving each problem-solving indicator based on Polya's framework.

Methods should be described with sufficient details to allow others to replicate and build on the published results. This section explains the research design, the reasons for the design, the research procedures applied, the population and research samples or participants, research instruments, data collection techniques, and data analysis techniques. The description should be in the past tense.

Results

The preparing for the experiment stage

In the research preparation stage, the researcher conducted a literature review, analysis, and discussions with several teachers and lecturers. In addition, the researcher also compiled a Student Activity Sheet (LAS) and designed a Hypothetical Learning Trajectory (HLT) to be applied in mathematics learning using the PMRI approach. The HLT design contains three main components: learning goals, learning activities, and hypothetical learning processes/conjectures (Gravemeijer & Cobb, 2006). The designed Hypothetical Learning Trajectory (HLT) is presented in Table 1.

Table 1. HLT design

Learning Goal	Learning Activities	Hypothetical Learning Processes/Conjectures
From the bicycle chain gear problem, students can identify the elements of a circle to solve the problem.	**Activity 1: Understanding Contextual Problems** 1. Students observe and explore the Augmented Reality presented regarding Bicycle Chain Gears	**Activity 1: Understanding Contextual Problems** 1. Students can write down the main problems and information based on the results of observations and Augmented exploration presented regarding Bicycle Gears in full.
	Activity 2: Modeling in a circular representation Students draw circles as representations of front and rear gears presented in Augmented Reality on the Activity Sheet.	**Activity 2: Modeling in a circular representation** Students can draw two circles as a representation of the front gear and the rear gear and connect the two circles with a straight line as a common tangent in accordance with AR in the Activity Sheet.
From the bicycle chain gear problem, students can solve problems related to the common tangent in two circles.	**Activity 3: Formulating assumptions and strategies** 2. Students write down their strategy predictions by paying attention to which parts will be used through Augmented Reality to solve the bicycle chain gear problem on the activity sheet. **Activity 4: Implementing the strategy** 3. Students apply the inner common tangent formula based on the information they have from Augmented Reality that has been observed regarding bicycle chain gears. **Activity 5: Reflection and Conclusion** 4. Students write conclusions from the results obtained after solving the bicycle chain gear problem using Augmented Reality on the activity sheet.	**Activity 3: Formulating assumptions and strategies** 2. Students write and explain which parts and the shape of the parts according to Augmented Reality which can be used to solve bicycle chain gear problems on the activity sheet. **Activity 4: Implementing the strategy** 3. Students can solve the total solar eclipse problem by applying the inner common tangent formula based on the information they have from Augmented Reality regarding the bicycle chain gears that have been observed. **Activity 5: Reflection and Conclusion** 4. Students can write conclusions and explain the results obtained after solving the bicycle chain gear problem using Augmented Reality on the activity sheet.

After the Student Activity Sheet and Hypothetical Learning Trajectory (HLT) were designed as shown in Table 1. Using the PMRI approach with the context of a solar eclipse and bicycle chain gears, the HLT was designed to guide students to achieve learning objectives through contextual activities utilizing Augmented Reality media. Before the small group trial, FGDs were conducted with teachers to revise the draft of the student activity sheet. The results of the revision are shown in Table 2. The results of this revision are in accordance with the characteristics of the PMRI approach that can be used in classroom learning

Table 2. FGD results

Suggestion	Revision
In the activity sheet identity, "Learning Achievements" are added to worksheets.	“Learning Outcomes” in worksheets have been added
In editorial number 12 for the sentence "Does it form a straight line?", it is emphasized again what context it is talking about.	The wording of number 12 for the sentence “Does it form a straight line?” has been clarified and changed to “Based on number 11, does the chain section form a tangent line?”
In the introductory editorial of activity 4 "By paying attention to the following formula, please relate it to the information you obtained from the problems in AR that have been given" just combine it with editorial number 13 "Based on the information you have, calculate the length of the part of the chain that does not encircle the gear"	The introductory words for activity 4 and number 13 have been combined into "Based on the tangent formula above and the information you have based on the augmented reality provided, calculate the length of the part of the chain that does not encircle the gear"

The design of the experiment stage

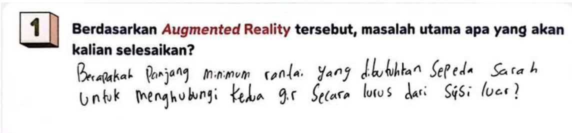
Pilot experiment

The pilot experiment was conducted for 80 minutes involving 9 students (3 groups) randomly selected from the 11th grade of SMA Srijaya Negara Palembang. The purpose of this cycle was to improve the HLT that had been designed previously in the initial stage through small class learning activities. The student activity sheet designed with the context of bicycle chain gears is shown in Figure 2.

Figure 2. Student activity sheet

Activity 1: Understanding Contextual Problems

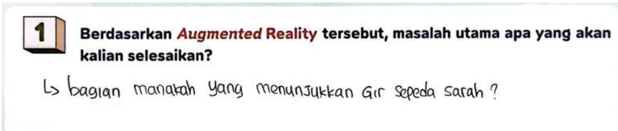
Students were asked to observe a contextual problem involving bicycle chain sprockets through Augmented Reality (AR) and identify the problem to be solved. Most students were able to record information from AR on the activity sheet, such as the distance and radius of the sprockets, thereby practicing their ability to gather information and determine the main problem, namely the minimum chain length required. However, one group gave an incorrect response, as shown in Figure 3.



1. Based on Augmented Reality, what is the main problem you will solve?

What is the minimum chain length required for Sarah's bicycle to connect the two sprockets straight from the outside?

(a)



1. Based on Augmented Reality, what is the main problem you will solve?

Which part shows Sarah's bicycle gears?

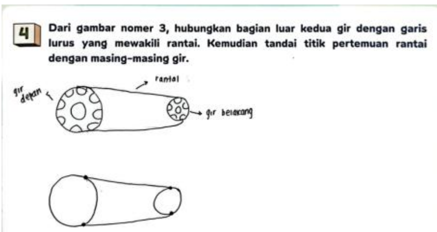
(b)

Figure 3. Student answers for activity 1 pilot experiment

In Figure 3(a) is the expected answer and 3(b) is the answer of 1 group that did not write the answer as expected because the answer given by the student was an instruction to go to the next display in Augmented Reality.

Activity 2: Modeling in a circular representation

Students sketched two circles (front and rear gears) and drew a chain that served as an external tangent. From the results of their work, all students were able to visualize the shape of a chain consisting of circular arcs and straight lines.as a tangent. This activity emphasizes visual representation skills.

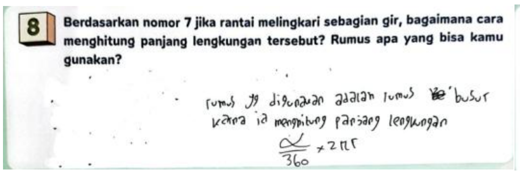


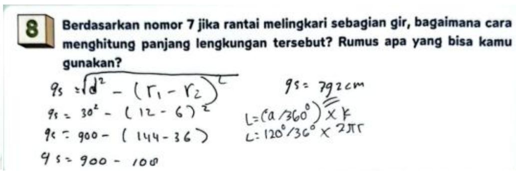
From image number 3, connect the outside of the two gears with a straight line representing the chain. Then mark the point where the chain meets each gear.

Figure 4. Student answers for activity 2 pilot experiment

Activity 3: Formulating conjectures and strategies

Students were asked to identify the chain components: an arc on the front sprocket, an arc on the rear sprocket, and a tangent line connecting them. Nearly all students answered correctly, but one group answered question 8 incorrectly, as shown in Figure 5.





8. Based on number 7, if the chain encircles part of the sprocket, how do you calculate the length of the arc? What formula can you use?

The formula used is the arc formula because it calculates the length of the arc. $\frac{\alpha}{360^\circ} \times 2\pi r$

8. Based on number 7, if the chain encircles part of the sprocket, how do you calculate the length of the arc? What formula can you use?

$gs = \sqrt{d^2 - (r_1 - r_2)^2}$
$gs = 30^2 - (12 - 6)^2$
$gs = 900 - (144 - 36)$
$gs = 900 - 100$
$gs = 792\ cm$
$L = (\frac{\alpha}{360^\circ}) \times k$
$L = \frac{120^\circ}{36^\circ} \times 2\pi r$

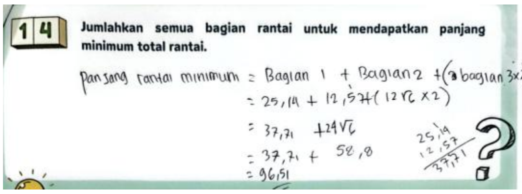
(a) (b)

Figure 5. Answers for activity 3 pilot experiment

Figure 5(a) shows the expected answer, while 5(b) is from a group that misread the question by giving the straight-line answer for the arc section, although they later corrected it. This activity trained students to apply mathematical structural strategies to real world problems.

Activity 4: Implementing the strategy

Students calculated the chain length using the arc formula for circular sections and the external tangent formula for straight sections. Some students correctly calculated the arc, but they were initially confused by the different symbols for the gear radii: "R" and "r" in AR versus "r_1" and "r_2" on the activity sheet. With teacher guidance, they were able to solve the problem correctly, as shown in Figure 6. This activity fostered procedural skills in mathematical modeling.



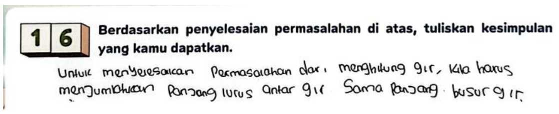
14. Add all the chain sections together to get the minimum total chain length.

Minimum length = Section 1 + Section 2 + (Section 3 x 2)
$= 25.14 + 12.57 + (12\sqrt{6} \times 2)$
$= 37.71 + 24\sqrt{6}$
$= 37.71 + 58.8$
$= 96.51$

Figure 6. Student answers for activity 4 pilot experiment

Activity 5: Carrying out reflection and conclusions

Students concluded the minimum chain length and noted possible alternative strategies. This activity emphasized concluding, verifying, and communicating results. All students stated there was no alternative method, as the solution required tangent and arc formulas. In concluding, they focused on adding arc and tangent lengths rather than restating the main problem, as shown in Figure 7.



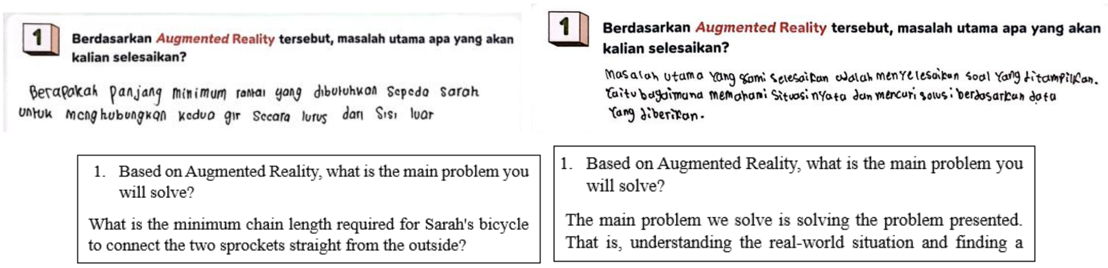
16. Based on the solution to the problem above, write down the conclusions you get.
To solve the problem of calculating the gear, we must add the straight length between the gears to the length of the gear arc.

Figure 7. Student answers for activity 5 pilot experiment

Teaching experiment

Before the Teaching Experiment, the activity sheet was revised based on the pilot results, such as adjusting the tangent formula symbols to "R" and "r" to match the AR display. The experiment was conducted in class XI.2 SMA Srijaya Negara Palembang with 35 students who worked together in groups consisting of 7 groups, each group consisting of 5 students.

Activity 1: Understanding contextual problems



(a) (b)

Figure 8. Student answers to activity 1 teaching experiment

Most groups identified key information from AR related to bicycle chain gears. Groups 2–6 wrote the main problem, gear sizes, and axle distance completely Figure 8(a). Groups 1 and 7 recorded only partial data: Group 1 omitted the main problem, and Group 7 miswrote the problem to be solved. This indicates varied problem-understanding skills, with most groups complete and some limited.

Activity 2: Modeling in a circular representation

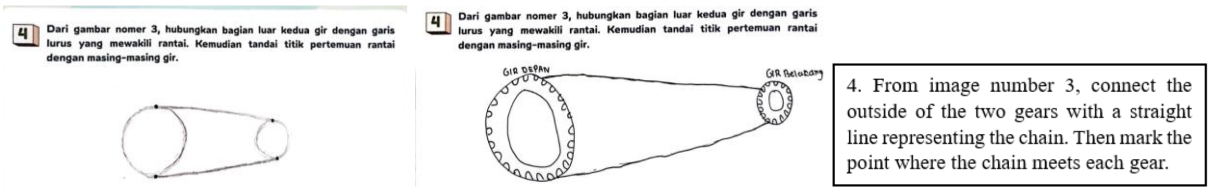
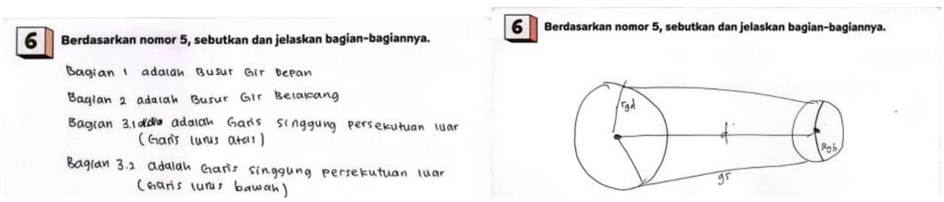


Figure 9. Student answers to activity 2 teaching experiment

All groups drew two circles to represent the gears. Groups 1–6 accurately visualized the chain as arcs and external tangents. While, Group 7 correctly depicted the chain, but the proportions of their sketch were unclear. This demonstrates that some students were capable, while others struggled to create an accurate visual representation.

Activity 3: Formulating conjectures and strategies



6. Based on number 5, name and explain the parts.
Section 1 is the front gear arc
Section 2 is the rear gear arc
Section 3.1 is the external common tangent (upper straight line)
Section 3.2 is the external common tangent (lower straight line)

(a)

6. Based on number 5, name and explain the parts.

(b)

Figure 10. Student answers for activity 3 teaching experiment

Groups 1, 2, 3, and 6 decomposed the chain into three parts: the front gear arc, the rear gear arc, and two tangents Figure 10(a). A new finding was that Group 4, besides making descriptive predictions, also formulated strategic predictions through visualizations Figure 10(b). To confirm understanding, the researcher conducted student interviews.

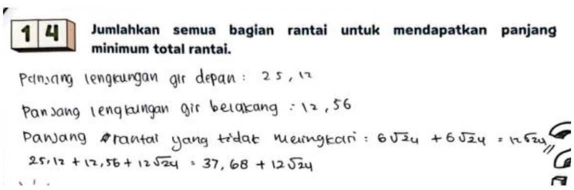
Researcher : Can you explain the picture you made?
Student : It shows two gears connected by a chain as seen in the Augmented Reality.
Researcher : Which parts are used to determine the minimum chain length?
Student : The curved parts of both gears and the two tangent lines.

Observations and interviews showed that students could identify the components needed to determine the minimum chain length.. Meanwhile, groups 5 and 7 were unable to explain it accurately. Then, with teacher guidance, both groups were able to understand and proceed to the next question correctly.

Researcher : Why is the answer blank? Try checking the Augmented Reality display again.
Student : I see three parts: the front gear arc, the rear gear arc, and the external tangents.
Researcher : Correct. These parts are used to determine the minimum chain length.

This conversation shows that after being given guidance by the teacher by paying attention again to the Augmented Reality provided, students who were initially confused about determining which part to use understood and were able to proceed to the next question correctly.

Activity 4: Implementing the strategy

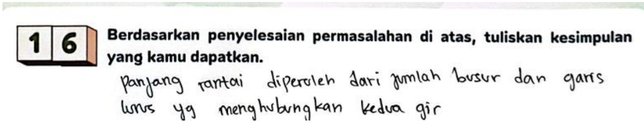

14 Jumlahkan semua bagian rantai untuk mendapatkan panjang minimum total rantai.
Panjang lengkungan gir depan : 25,12
Panjang lengkungan gir belakang : 12,56
Panjang rantai yang tidak melingkari : $6\sqrt{24} + 6\sqrt{24} = 12\sqrt{24}$
$25,12 + 12,56 + 12\sqrt{24} = 37,68 + 12\sqrt{24}$

14. Add all the chain sections together to get the minimum total chain length.
Length of the front sprocket arch = 25,12
Length of the rear sprocket arch = 12,56
Length of the chain that does not circle = $6\sqrt{24} + 6\sqrt{24} = 12\sqrt{24}$
$25,12 + 12,56 + 12\sqrt{24} = 37,68 + 12\sqrt{24}$

Figure 11. Student answers for activity 4 teaching experiment

Based on Figure 11 in calculating the strategy, all groups were able to determine the solution. This shows that procedural skills varied: some groups solved the problem down to the simplest solution, while others solved it without simplifying the root. However, overall, all groups were able to determine the minimum chain length needed to connect the two bicycle sprockets.

Activity 5: Carrying out reflection and conclusions



16. Based on the solution to the problem above, write down the conclusions you get.
The length of the chain is obtained from the number of arcs and straight lines connecting the two gears.

Figure 12. Student answers for activity 5 teaching experiment

Most groups gave no further strategies and did not conclude their problem-solving. One group did conclude, but only by explaining that the chain length was the sum of arcs and tangents. They could not mathematically summarize results, connect them to the main problem, or justify their reasoning. Thus, the researchers conducted interviews.

Researcher : *What is the main problem and the result you obtained?*
Student : *The problem is finding the minimum chain length, which is 108.1 cm, calculated by adding the arcs and the tangents.*
Researcher : *Why didn't you write the conclusion on the worksheet?*
Student : *I understood it, but I was in a hurry and didn't write it down.*

Based on observations and interviews, several groups were able to summarize their findings and relate them to the contextual issues discussed, but they didn't write them down on the activity sheet because they were in a rush to answer. Overall, their ability to summarize and communicate their findings was limited, with only a few groups completing the process.

Written test

The test was given after the teaching experiment to evaluate how the designed activities supported students' problem-solving abilities. It consisted of six questions on circle tangents, and the indicators of problem-solving skills from the results are shown in Table 3.

Table 3. Emergence of problem solving indicators

Indicator	Question					
	1	2	3	4	5	6
Understanding the problem	71%	77%	100%	97%	83%	34%
Create a solution plan	100%	100%	100%	100%	86%	20%
Implementing the solution plan	100%	100%	100%	100%	86%	20%
Evaluating solutions	17%	51%	17%	47%	14%	0%

Table 4. Data analysis test

Variables	*Mean*	*Standard Deviation*	*Mean Difference*	*Standard Deviation Difference*
Pre-test	19.37	11.98	-47.14	2.6
Post-test	66.51	9.38		

Based on the occurrence of Polya's problem-solving indicators across the six questions, an overall pattern indicates that students demonstrated strong problem-solving abilities on tasks with low to moderate levels of difficulty (questions 1–4), with achievement percentages ranging from 71% to 100%. However, a considerable decline was observed on tasks with higher complexity (questions 5–6), particularly on question 6, where the indicator of understanding the problem reached only 34%. This pattern suggests that students were generally able to solve easier problems but experienced difficulties as cognitive demands increased.

These findings are supported by descriptive statistics from the written test, which show a substantial improvement in students' mean scores from the pre-test (M = 19.37; SD = 11.98) to the post-test (M = 66.51; SD = 9.38). The increase in mean scores indicates an overall improvement in students' problem-solving abilities after the implementation of the learning trajectory, while the decrease in standard deviation in the post-test suggests that students' performance became more consistent and evenly distributed.

When examined by indicator, students showed the strongest performance in devising and implementing solution strategies, with 100% achievement in questions 1–4, followed by a decrease to 86% in question 5 and 20% in question 6. In contrast, the evaluation of solutions consistently emerged as the weakest indicator across all questions, with relatively low percentages ranging from 17% to 51% in questions 1–4, declining to 14% in question 5, and reaching 0% in question 6. This indicates that students rarely engaged in rechecking their work or writing conclusions, even when they were able to obtain correct computational results. Based on interview and observation data, this tendency was influenced by students' habitual assumption that obtaining a correct answer was sufficient and by their tendency to work hastily. Overall, these results indicate that the PMRI-based learning trajectory supported by Augmented Reality was effective in enhancing students' abilities to understand problems and to devise and implement solution strategies, as reflected in the increased mean scores and high achievement percentages in the initial problem-solving indicators. Nevertheless, the evaluation stage remained a consistent weakness, particularly when students were faced with more complex problems.

The retrospective analysis stage

The retrospective analysis focused on comparing the Hypothetical Learning Trajectory (HLT) with the Actual Learning Trajectory (ALT) observed during classroom implementation. Overall, the results indicate a general alignment between the designed learning trajectory and students' actual learning processes, although several discrepancies were identified at different stages of the learning activities.

Table 5 summarizes the key alignments and discrepancies between the HLT and ALT across each phase of the learning trajectory, including problem understanding, situation modeling, strategy building, strategy implementation, and reflection. A comparison of HLT and ALT for circle learning in the context of bicycle chain gears is shown in Table 5 below.

Table 5. Comparison of HLT and ALT circle learning

Learning Stage	HLT	ALT
1. Problem Understanding	1. Students identify all relevant information and clearly state the main problem based on AR exploration.	1. Most students identified important information from AR, but some did not write the main problem completely.

Learning Stage	HLT	ALT
2. Situation Modeling	2. Students draw proportional circle representations and tangent lines based on the given context.	2. Almost all groups drew circles and tangents, although some sketches were not proportional.
3. Strategy Building	3. Students identify arc and tangent components needed to solve the problem.	3. Most groups identified the correct components, but a few made initial errors and required teacher guidance.
4. Strategy Implementation	4. Students apply arc and tangent formulas consistently to calculate chain length.	4. All groups attempted calculations, though some were confused by different symbols used in AR and worksheets.
5. Reflection and Conclusion.	5. Students write complete conclusions linking results to the main problem.	5. Only a few groups wrote complete conclusions; most listed solution steps without linking them to the problem context.

Retrospective analysis showed that the designed Hypothetical Learning Trajectory (HLT) largely aligned with the Actual Learning Trajectory (ALT) observed in classroom implementation, although some discrepancies were still identified. The initial stages of the learning trajectory: problem understanding, situation modeling, and strategy building were generally implemented as expected. Most students were able to identify important information through Augmented Reality (AR), draw circles and tangents, and determine relevant arc and tangent components, although minor discrepancies occurred in students' written articulation, proportional representations, and the use of symbols. Greater discrepancies were observed in the reflection and conclusion stage, where only a few groups wrote complete conclusions, while most students focused on outlining solution steps without explicitly linking them to the main problem. Overall, these findings indicate that AR effectively supported students in understanding, planning, and implementing solution strategies, while evaluation and reflection skills remained the weakest aspect of the learning trajectory.

Discussion

The findings indicate that the learning trajectory designed through the Hypothetical Learning Trajectory (HLT) largely aligned with the Actual Learning Trajectory (ALT) observed in classroom implementation. Students generally followed the intended sequence of understanding contextual problems, modeling situations using circles, and developing and implementing solution strategies. This supports the view that HLT functions as a hypothetical guide that is tested and refined through classroom practice, resulting in an ALT that reflects students' actual learning processes (Gravemeijer & Cobb, 2006).

Analysis of Polya's problem-solving indicators across six test questions revealed that students demonstrated strong performance in understanding problems as well as in devising and implementing solution strategies, particularly on tasks with low to moderate levels of difficulty. In contrast, the evaluation stage consistently emerged as the weakest indicator across almost all questions, with performance declining further as task complexity increased. This pattern suggests that while students were proficient in procedural and strategic stages of problem solving, they were less accustomed to checking, justifying, and reflecting on their solutions (Serang et al., 2025; Syamsuddin, 2020).

These results reflect the role of Augmented Reality (AR)-based worksheets in supporting students' problem-solving processes. AR assisted students in visualizing geometric concepts and developing solution strategies, as evidenced by the high achievement percentages in planning and implementation indicators (Hou, 2022; Nindiasari et al., 2024). This finding is consistent with previous research indicating that visual and interactive representations can bridge abstract mathematical concepts with concrete understanding, thereby facilitating conceptual connections and procedural competence during problem-solving activities (Fokuo et al., 2023; Li et al., 2022; Rahmadi et al., 2024).

However, the persistent weakness in the evaluation stage indicates that the learning trajectory design requires further strengthening in the reflection component. In line with prior studies, limited reflective skills may be influenced by learning habits that prioritize obtaining correct numerical answers over examining solution processes or validating results (Pangaribuan et al., 2025; Syamsuddin, 2020). Consequently, students may perceive problem solving as complete once a correct answer is reached, without recognizing the importance of justification and reflection, a tendency that has also been reported in mathematics learning contexts emphasizing procedural achievement (Yayuk et al., 2020).

From a learning trajectory perspective, these findings highlight the need to explicitly embed reflective activities within subsequent HLT designs. Reflection should be treated as an integral part of the problem-solving process rather than an optional final step. Such activities may include guided group discussions, teacher-prompted reflective questions, and written conclusion tasks that require students to connect their solutions to the original problem context (Syamsuddin, 2020). By explicitly integrating reflection into the learning trajectory, students' evaluation skills can be strengthened alongside their procedural and strategic abilities.

Overall, this study indicates that PMRI-based learning supported by Augmented Reality effectively enhances students' abilities to understand problems and carry out solution strategies. Comparable findings have been reported in prior studies on PMRI and technology-supported mathematics learning, which highlight the role of contextual learning environments and digital tools in fostering students' engagement and problem-solving performance (Komariyatiningsih et al., 2023; Meryansumayeka et al., 2022; Zulkardi et al., 2020). Nevertheless, to foster more holistic problem-solving skills, greater instructional emphasis is required at the evaluation stage. Providing systematic scaffolding for reflection may enable students to develop a more complete problem-solving process that encompasses understanding, planning, implementing, and evaluating solutions (Serang et al., 2025)

Conclusion

A PMRI-based learning trajectory supported by Augmented Reality on circle material was designed through a sequence of contextual and AR-assisted activities. The trajectory guided

Grade XI senior high school students from understanding real-world circle-related problems, modeling situations using circle concepts, to progressively developing and implementing solution strategies. This design enabled students to construct mathematical concepts gradually through meaningful contexts and dynamic visual representations.

The findings indicate that the PMRI-based learning trajectory supported by Augmented Reality played an important role in supporting students' problem-solving skills. The learning trajectory effectively facilitated students' abilities to understand problems, devise solution plans, and carry out solution strategies. However, students' performance in the evaluation and reflection stage remained relatively weak, indicating that reflective problem-solving skills require explicit instructional scaffolding and do not develop automatically through technology-supported learning.

From a theoretical perspective, this study contributes to learning trajectory research by demonstrating how PMRI principles and augmented reality can be systematically integrated to support students' problem-solving processes in geometry learning. Practically, the findings provide guidance for mathematics teachers in designing AR-supported PMRI instructional activities that emphasize not only procedural and strategic problem solving but also structured reflection and solution evaluation.

This study was limited by the relatively short duration of implementation and its focus on a single geometry topic. Future research is recommended to investigate longer-term implementations of AR-supported learning trajectories and to develop instructional designs that more explicitly strengthen students' reflective and evaluative problem-solving skills.

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Declarations

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