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# Developing a built-in definition of fraction manipulative to enhance students' conceptual understanding

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**Abstract**

This study addresses the challenge of students limited conceptual understanding of fractions by developing a physical manipulative called the *Built-in Definition of Fraction* (BDF). Using design research developmental studies, data were collected through written tests, questionnaires, and interviews. A total of 41 fifth-grade students participated in the study. The BDF and its accompanying student worksheet integrate dual visual-symbolic representations, that link each concrete action to its corresponding fraction notation and scaffold students in constructing their own understanding of fractions, offering a new approach to using fraction manipulatives. Validation results show that the manipulative is valid, practical, and effective in supporting student understanding of basic fraction concepts, equivalent fractions, and same-denominator addition, even with minimal teacher guidance. Student performance was high across visual-concrete tasks. However, a noticeable gap emerged in unlike-denominator addition between visual concrete performance (88%) and formal-symbolic procedures (77%). This finding highlights the need for additional scaffolding to help students transition more smoothly from concrete representations to abstract symbolic reasoning.

**Keywords:** built-in definition of fractions; conceptual understanding; fraction learning; instructional media development; physical manipulatives; realistic mathematics education; square model

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## Introduction

Understanding fractions is widely recognized as one of the most challenging topics for students (Gabriel et al., 2013; Sari et al., 2024; Singh et al., 2021; Unaenah et al., 2024). Difficulties arise not only in grasping the basic meaning of fractions but also in performing operations and solving problems involving them (Hariyani et al., 2022). Conceptually, fractions differ from whole numbers because they represent relationships between quantities rather than discrete counts. This shift from viewing numbers as whole units to interpreting the numerator and denominator relationally tends to confuse students (Braithwaite et al., 2018; Lortie-Forgues et al., 2015; Siegler et al., 2011), often leading to a whole-number bias as described by Ni and Zhou (2005).

Fractions can be represented across multiple interpretations, including as part-whole relationships, division, measurement, ratio, and operator forms (Cramer et al., 2020; Krause et al., 2016; Sevier et al., 2022; Simon et al., 2018; Wilkins & Norton, 2018). Among these, part-whole relationships are often considered an effective starting point for developing fraction concepts (Streefland, 1991). However, students' exposure to diverse part-whole situations is frequently limited. In the Indonesian context, for example, students typically encounter only familiar terms such as "half" (*setengah*) or "quarter" (*seperempat*), without engaging with broader representations that could deepen their intuition (Unaenah et al., 2024). As a result, many students struggle to interpret symbols, compare fractions, recognize equivalence, or apply operations meaningfully (Braithwaite et al., 2018; Lortie-Forgues et al., 2015; Ni & Zhou, 2005; Siegler et al., 2011).

Furthermore, Kilpatrick, Swafford, and Findell (2013), in their framework of mathematical proficiency, position conceptual understanding as one of the five interrelated strands supporting mathematical competence, alongside procedural fluency, strategic competence, adaptive reasoning, and productive disposition. They define conceptual understanding as comprehension of mathematical concepts. Without this understanding, students tend to treat fractions as two whole numbers separated by a bar and rely on rote procedures when operating with fractions, rather than developing meaningful understanding (Lamon, 2020; Siegler et al., 2011).

Physical manipulatives have long played an essential role in mathematics education by providing hands-on tools for students (Abarquez, 2020; Merkel et al., 2025; Parungao, 2021; Zwanch & Mullins, 2025). Commonly used manipulatives such as fraction circles and fraction bars help students visualize and represent fractions (Maboya et al., 2020; Siller & Ahmad, 2024; Wilkie & Roche, 2023), and a substantial body of research has demonstrated their effectiveness in supporting early fraction understanding (Carbonneau et al., 2013; Vessonen et al., 2021). Within the Realistic Mathematics Education (RME) perspective, manipulatives serve as bridges between concrete experience and abstract mathematical reasoning, helping students construct conceptual meaning through structured activity (Gravemeijer, 1994; Van den Heuvel-Panhuizen, 2003; Abarquez, 2020; Carbonneau et al., 2013).

Circle models, bar models, and number lines remain the most widely used representations in fraction instruction (Amo-Asante & Bonyah, 2023; Bruce et al., 2023; Monson et al., 2020;

Powell, 2023). These representations provide concrete supports that help students visualize part–whole relationships, reason about equivalence, and make sense of fraction magnitude (Bruce et al., 2023; Resnick et al., 2023). However, research also indicates that relying on only one or two visual models can constrain students' conceptual development. Exposure to diverse and structurally distinct representations is therefore essential for cultivating robust fraction understanding (Cramer et al., 2002; Lamon, 2020). In particular, the circle model introduces perceptual difficulties when students attempt to divide it into equal parts, making the development of partitioning strategies more challenging (Keijzer & Terwel, 2001; Moss & Case, 1999). Square models, by contrast, offer a more geometrically regular structure that supports accurate and equitable partitioning.

Despite the extensive use of fraction manipulatives, limited studies have examined square-based physical models or manipulatives that explicitly embed the definition simultaneously. This presents a conceptual gap: students often manipulate objects visually without connecting these actions to the underlying mathematical structures the objects are intended to represent. Building on this premise, the present study investigates the potential of a square-based area model embedded in the Built-in Definition of Fraction (BDF) manipulative. The use of this manipulative is integrated with a structured instructional sequence in the form of student worksheets designed to guide students in understanding the concept of fractions, equivalent fractions, and fraction addition by making explicit connections between their actions with the manipulative and the corresponding mathematical structure. Therefore, this study aims to develop and validate the BDF manipulative and its accompanying student worksheets in terms of their validity, practicality, and effectiveness in fostering students' conceptual understanding of fractions.

## Methods

This study employed a design research method, combining development studies (Van den Akker et al., 2006) and Tessmer's (1993) formative evaluation framework. The aim was to develop an instructional manipulative that is valid, practical, and demonstrates potential effects in supporting students' conceptual understanding. The product developed in this study is referred to as the *Built-in Definition of Fraction* (BDF). In addition, a set of Student Worksheets was designed to guide students in learning fraction concepts using the manipulative and to serve as complementary materials during the learning activities.

The study involved a total of 41 fifth-grade elementary students. Of these, 32 students served as the main participants in the field tests, while an additional 9 students from other classes participated in the one-to-one and small group stages. The participating class was selected through purposive sampling because the classroom teacher had previously participated in training on the use of the BDF manipulative. During classroom implementation, teacher intervention was intentionally minimized to allow students' interactions with the manipulative and the worksheets to emerge naturally. The fifth-grade students who participated in the study had previously learned basic fraction concepts and equivalent fractions in earlier grades. However, fraction addition, particularly involving unlike denominators was new to them.

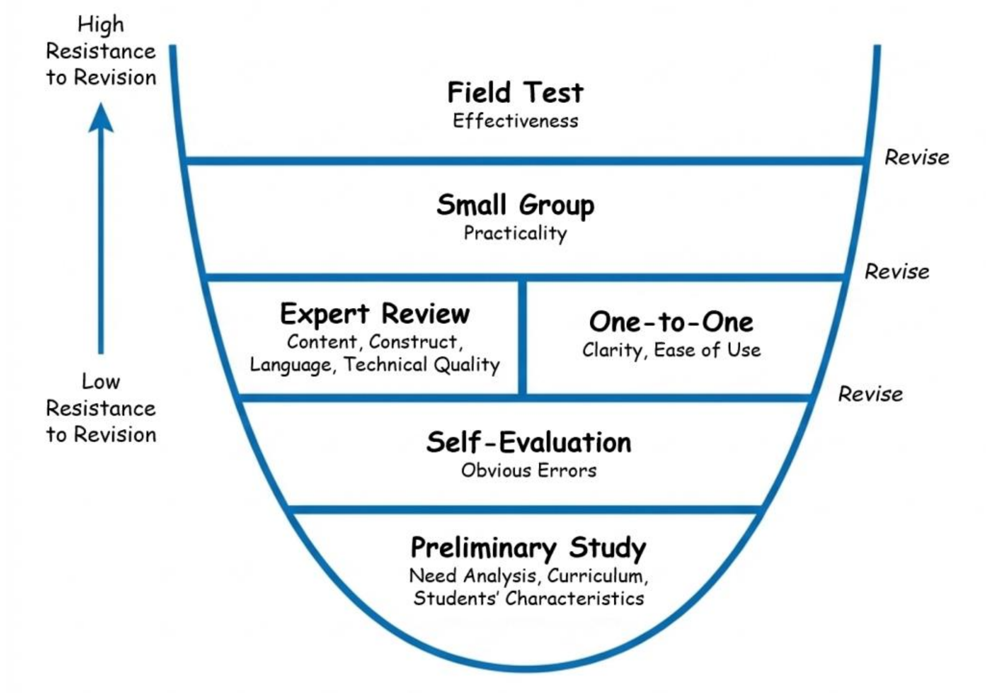


**Figure 1.** Stages of the design research and formative evaluation

Figure 1 illustrates the stages of the study. A validation form was used to evaluate the content, construct, language, and technical quality of the manipulative and student's worksheet through expert review. The practicality was assessed using student questionnaires administered after the small group. The effectiveness of the materials in supporting students' conceptual understanding was further examined through a field test and supplemented by interviews with selected students, providing deeper insight into students' learning experiences.

Consistent with design research principles and formative evaluation frameworks, the validity and practicality of the BDF manipulative and student worksheets were classified based on mean scores derived from a Likert scale (1 for strongly disagree to 4 for strongly agree).

**Table 1.** Categories of validity and practicality

| Mean Score Range | Category of Validity | Category of Practicality |
|---|---|---|
| 3.25 – 4.00 | Very valid | Very Practical |
| 2.50 – 3.24 | Valid | Practical |
| 1.00 – 2.49 | Less Valid | Less Practical |

## Results

### Preliminary study and self-evaluation

A comprehensive review of the curriculum, the developmental characteristics of elementary school learners, and the essential features of fraction-related instructional media was conducted during the preliminary phase. In alignment with the *Kurikulum Merdeka*, the teaching of fractions in elementary schools is organized across three developmental phases. Phase A (Grades 1–2) introduces students to fractions through multiple forms of representation. Phase B (Grades 3–4) engages students in comparing fractions and developing an understanding of equivalent fractions. Phase C (Grades 5–6) focuses on arithmetic operations involving fractions. Given the wide scope of fraction learning, this study focuses specifically on fraction addition, allowing for a more in-depth and meaningful exploration of how students develop understanding in this topic. To engage meaningfully with fraction addition, students need a strong conceptual grounding in both basic fraction concepts and equivalent fractions. For this reason, the learning trajectory in this study is intentionally designed to help students build these prerequisite concepts first.

According to Piaget's theory of cognitive development, elementary school students are in the concrete operational stage (ages 7–11), during which they are capable of logical reasoning, but primarily with objects and situations that are tangible, visible, or directly manipulable (Piaget, 2013). Consistent with this perspective, Tall proposed that students' mathematical understanding develops through the embodied, symbolic, and formal stages (Tall, 2008). Elementary students typically operate at the embodied stage, where activity-based and concrete experiences support the construction of mathematical concepts. Instructional media or manipulatives can serve as a bridge, enabling students to form a robust *concept image*, which is essential for developing a more formal *concept definition* (Tall, 2014).

In line with these theoretical perspectives, this study developed a fraction manipulative (BDF) which employs a square model as its primary representation. This square model in the manipulative maintains consistent unit sizes and ensures that fractional parts originate from the same whole, allowing students to develop an understanding of part–whole relationships (Fisher, 2009). Additionally, this manipulative enables students to manipulate and compare parts directly, aligning with the concrete reasoning emphasized by Piaget and the embodied stage described by Tall. The principles of embodied cognition further suggest that physical interaction with learning materials strengthens the formation of mental representations and supports deeper conceptual understanding (Nemirovsky et al., 2013).

**Figure 2.** Prototype 1 of the BDF manipulative

Building on these theoretical foundations, the BDF manipulative developed in this study connects students' concrete actions with fraction concepts and operations. The relationship between actions performed with the manipulative and the corresponding conceptual ideas is summarized in Table 2.

**Table 2.** Relationship between concrete actions and fraction concepts

| Concrete Actions on the Manipulative | Fraction Concepts and Operations |
|---|---|
| Counting fractional pieces (unit fractions) and comparing them to the total within the same color | Representing fractions as parts of a whole |
| Comparing the sizes of fractional pieces | Ordering fractions |
| Identifying fractional pieces of equal size | Recognizing equivalent fractions |
| Combining fractional pieces of the same color | Adding fractions with the same denominator |
| Swapping fractional pieces with others of equal size but different colors | Establishing a common denominator as a prerequisite for fraction addition |

This design reflects the concrete–representational–abstract (CRA) approach (Walle et al., 2019), ensuring that students' hands-on interactions with the manipulative provide a foundation for developing formal conceptual understanding and procedural fluency in fractions. By integrating concrete actions with visual and symbolic representations, the BDF manipulative offers a coherent pathway for students to progress from embodied experiences to formal mathematical reasoning.

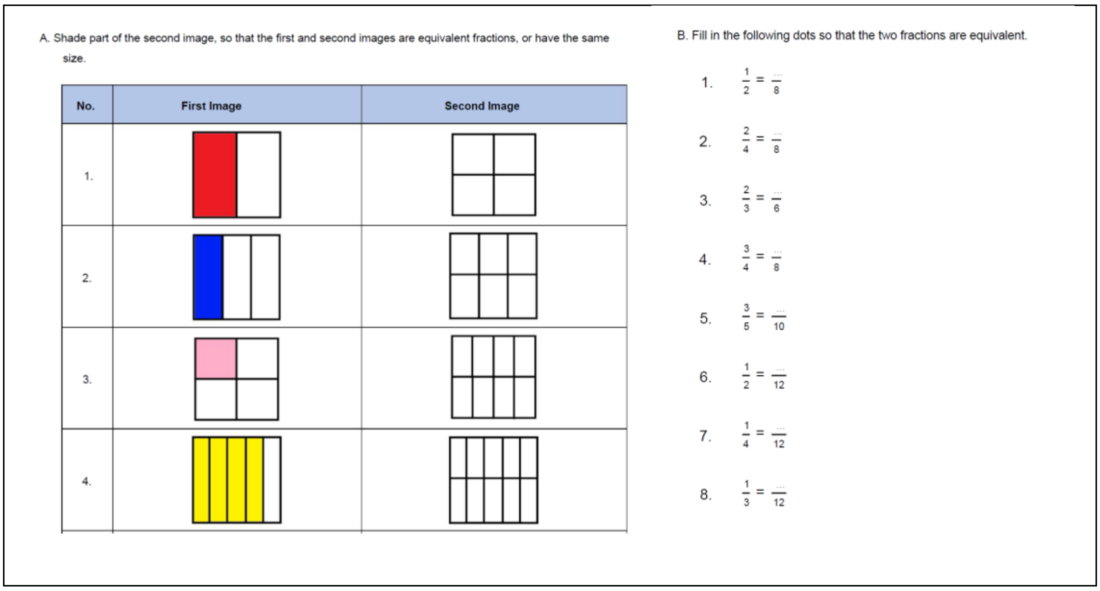


**Figure 3**. Prototype 1 of the worksheet on the topic of equivalent fraction

## Expert review

The validation of the BDF manipulative and the accompanying worksheets was conducted using a validation form completed by five experts. The validation focused on four main aspects: 1) the compatibility of the manipulative and students worksheets in supporting conceptual understanding, 2) the construction quality of the manipulative, 3) the correctness and the clarity of the language used in the worksheets, and 4) the potential ease of use of both the manipulative and worksheets.

**Table 3.** Validity scores

| Validation Aspects | Average Score | Category |
|---|---|---|
| Compatibility of the manipulative and students worksheets to support conceptual understanding, including five indicators:<br>1) compatibility of the manipulative for understanding part-whole relationships<br>2) compatibility for supporting understanding of equivalent fractions<br>3) compatibility for supporting understanding of fraction addition<br>4) alignment of worksheet instructional support with the curriculum<br>5) alignment with learning outcomes | 3.84 | Very valid |
| Construction quality of the manipulative, including three indicators:<br>1) visually appealing color selection<br>2) proportional design<br>3) use of durable materials | 3.93 | Very valid |
| Correctness of the language used in the worksheet, including three indicators:<br>1) appropriate word choice<br>2) clarity of instructional commands<br>3) accurate use of mathematical terms and symbols | 3.87 | Very valid |
| Potential ease of use of the manipulative and worksheets, including four indicators:<br>1) ease of grasping and storing the manipulative<br>2) strong potential for ease of use by students<br>3) strong potential for ease of use by teachers<br>4) potential to promote active student engagement | 3.85 | Very valid |

The expert validation results indicate that the developed manipulative and worksheets are highly valid. However, the experts also provided several recommendations, including ensuring consistent use of the terms "part" and "whole," and incorporating more elements that encourage active engagement within the worksheets to strengthen their compatibility with the manipulative.

## One-to-one

In line with the expert review, the manipulative and worksheets were pilot-tested with three students who were not part of the main study. The selected students were fifth graders representing high, medium, and low academic ability levels. Based on the pilot test, students in the medium and low ability categories experienced difficulties when completing formal-symbolic tasks. They performed relatively well on tasks involving visual-concrete representations but encountered challenges when working with formal tasks. In the first prototype of the worksheet, the visual representation tasks were differentiated from the symbolic tasks, as shown in Figure 3. In the visual representation tasks, the worksheet provided only the visual form of equivalent fractions, while in the symbolic tasks, all problems were presented in symbolic form, without accompanying visual.

According to the study's conjecture, this separation should not have created differences in performance, because students were expected to develop their understanding through concrete activities and visual representations and then apply the same reasoning to solve symbolic problems using the BDF manipulative. Through repeated engagement with the manipulative, students were expected to use the same way to solve the symbolic tasks. However, during this stage, high-ability students were able to solve formal-symbolic tasks, but the strategy did not involve the use of the BDF manipulative. An excerpt from an interview with a high-ability student illustrates this point.

***Observer:*** *"How did you know that 1/2 = 4/8?"*
***Student 1:*** *"I multiplied everything by 4."*
***Observer:*** *"What did you multiply by 4?"*
***Student 1:*** *"2 × 4 = 8, so I also multiplied 1 by 4."*

This finding suggests that although high-ability students can successfully perform formal-symbolic manipulations, they do not automatically integrate the BDF manipulative into their problem-solving strategies. This indicates an area where further instructional scaffolding is needed. In contrast, the transition from visual-concrete tasks to formal-symbolic tasks was unsuccessful for medium- and low-ability students. They were able to solve visual-concrete tasks but were unable to solve the formal-symbolic problems.

Based on the findings from the one-to-one trials and expert review, Prototype 1 was revised with attention to the following aspects.

**Table 4.** Revision decisions for the manipulative and worksheets

| Aspect | Finding | Revision |
|---|---|---|
| Ease of manipulative use | Students could operate the manipulative easily | No revision |
| Clarity | Instructions were clear and easy to follow | No revision |
| Compatibility between the manipulative and worksheets | Students encountered difficulty transitioning from visual-concrete to formal-symbolic tasks | The worksheets were revised to facilitate dual representation, incorporating both visual and symbolic formats |

These findings formed the basis for designing worksheets that are compatible with the manipulative and that effectively support conceptual understanding through dual representations. In Prototype 2, this representational bridge was not optimally supported, resulting in students performing well on visual–concrete tasks but struggling with formal–symbolic ones. Therefore, in Prototype 3, dual representation, visual and symbolic, was explicitly incorporated to support students' transition toward more abstract mathematical reasoning.

The dual representation was introduced in the visual–concrete tasks by adding a column labeled "symbol," which denotes the fraction notation corresponding to the actions performed using the manipulative. With this column, each step of visual manipulation, such as comparing the areas of colored parts, is directly linked to its symbolic expression. This structure is designed to help students bridge concrete actions with formal symbolic structures, enabling coordination between representations to occur more naturally. Consistent with Moyer (2001) and Uttal et al. (1997), manipulatives are effective only when students can connect the concrete actions they perform with the mathematical ideas those actions represent. In this way, students are not merely moving or manipulating objects, but they are actively constructing meaningful and coherent conceptual understanding.

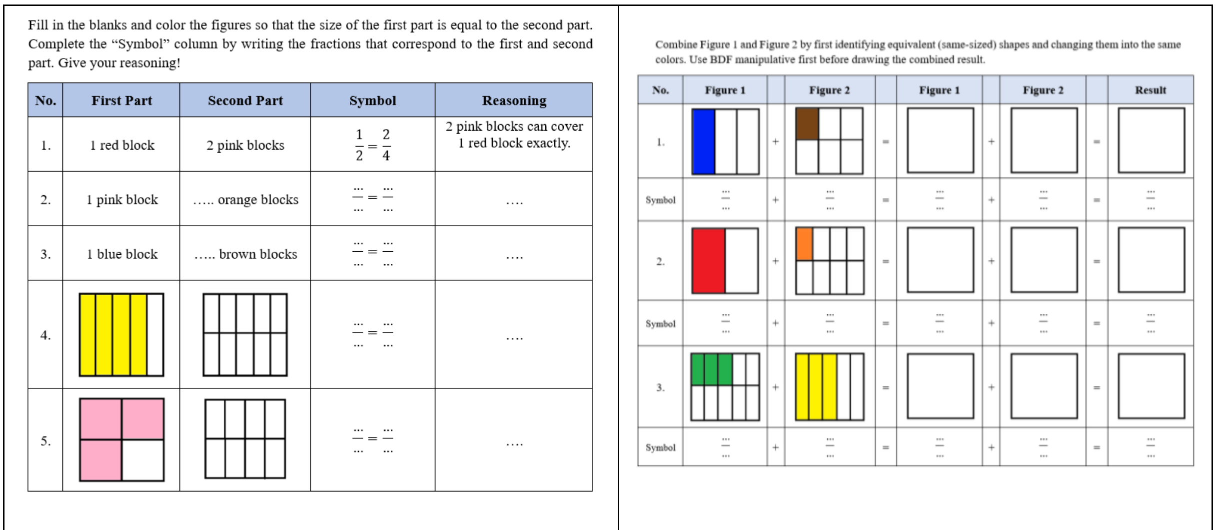


**Figure 4.** (a) Prototype 2 of the worksheet on the topic of equivalent fractions (b) Prototype 2 of the worksheet on the topic of fraction addition

## Small group

Prototype 2 was tested in a small group consisting of six students who were not part of the main study. During this stage, several typographical errors were identified and subsequently corrected. It was also observed that the allocated time was insufficient for students to complete all the tasks. As a result, the number of tasks was reduced in Prototype 3, which was then prepared for the field test. The practicality of the manipulative and worksheets was also assessed at this stage. The results of the practicality evaluation, by the questionnaire responses, are presented in the following table.

**Table 5.** Practicality test results

| Questionnaire Item | Mean | Category |
|---|---|---|
| The colors of the manipulative are very appealing and motivate me to learn | 4.00 | Very practical |
| The design is neat, and all components are well organized | 4.00 | Very practical |
| The manipulative helps me understand the topic | 3.83 | Very practical |
| I can solve fraction problems faster with the support of the manipulative | 3.83 | Very practical |

| Questionnaire Item | Mean | Category |
|---|---|---|
| I am able to use the manipulative independently without difficulty | 3.83 | Very practical |
| The part of the manipulative are easy to manipulate and move around | 4.00 | Very practical |
| Average | 3.92 | Very practical |

## Field test

Prototype 3 represents the final version of the worksheets developed in this study. The Prototype 3 worksheets and the BDF manipulative were subsequently tested in a field test involving 32 fifth-grade students. The purpose of this field test was to examine the potential effects of the manipulative and worksheets in supporting students' conceptual understanding. The results of the field test are presented in the following diagram.

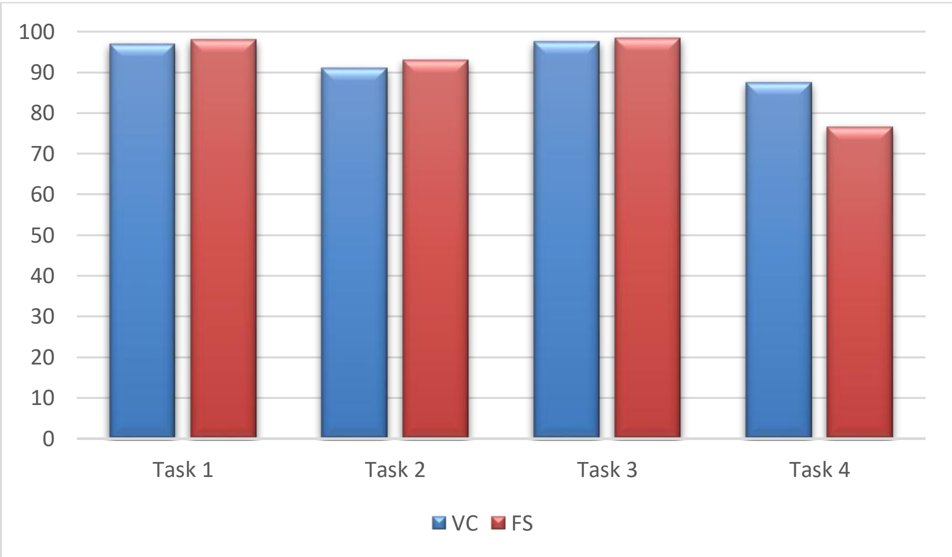


**Figure 5.** Field test result

The results were organized and analyzed across four tasks: (1) Task 1: basic fraction concepts, (2) Task 2: equivalent fractions, (3) Task 3: fraction addition with the same denominators, and (4) Task 4: fraction addition with unlike denominators. For each topic, students' performance was further categorized according to two stages of learning activities: the visual-concrete stage (VC), where tasks were supported with concrete visualizations of the BDF manipulative, and the formal-symbolic stage (FS), where students engaged with more abstract, symbolic representations of fractions. This framework allowed for a detailed analysis of students' conceptual progression, from hands-on exploration to formal mathematical reasoning.

The results from the field test indicate that students generally performed very well on the first three tasks, with scores consistently ranging from 91% to 99% for both Visual-Concrete (VC) and Formal-Symbolic (FS) tasks. Task 1 and Task 3, in particular, showed almost identical performance between the two representations, suggesting that students were able to successfully translate their understanding of fractions between visual-concrete and symbolic formats for these tasks. A slight decrease in performance is observed in Task 2, though students still achieved high scores, indicating that the tasks were manageable and the BDF manipulative effectively supported conceptual understanding at both representation levels. Task 4, however, exhibited the lowest scores and the largest disparity between VC and FS tasks. While students scored approximately 88% on the VC task, the FS task dropped to around 77%, indicating that students encountered greater difficulty when required to solve formal-symbolic problems. This finding aligns with previous observations from one-to-one trials and small-group tests, where students, particularly those with medium and low academic ability, struggled to apply the manipulative in symbolic problem-solving.

## Discussion

Across the iterative design cycle of the BDF manipulative, several key characteristics that enhance its instructional effectiveness became increasingly evident. Foremost among these is the manipulative's capacity to provide integrated dual representations, in which every concrete action performed on the physical model is immediately connected to its symbolic fraction notation. This connection facilitates a deeper conceptual understanding of fractions by bridging the gap between tangible actions and abstract mathematical symbols (Bruce et al., 2023; Merkel et al., 2025; Wilkie & Roche, 2022). Students develop their understanding of part–whole relationships by comparing fractional pieces with the whole unit of the same color. To explore equivalent fractions, students identify fractional pieces of different colors that are equal in size. For fraction addition, students either combine fractional pieces directly or exchange them for pieces of equivalent size in other colors to establish common denominators. Through these coordinated actions, students actively construct symbolic meaning from their interactions, reinforcing the interpretive link emphasized in Bruner's enactive–iconic–symbolic framework and supported by representational-bridging research (Ainsworth, 2006; Uttal et al., 1997).

The second distinguishing feature is the manipulative's high level of practicality and independent usability. Students were able to handle, rearrange, and compare fractional pieces with ease, demonstrating that the manipulative functions as a tool for thinking. Previous studies find that manipulatives must be simple enough to operate but robust enough to support meaningful reasoning (Akuom & Greenstein, 2022; Donovan & Alibali, 2021; Yuan et al., 2021). In practice, the BDF components facilitated exploratory learning and encouraged students to engage independently with the tasks, aligning with Piaget's view that concrete interaction plays an important role in conceptual development and with research showing that embodied activity enhances mathematical reasoning (Nemirovsky et al., 2013).

A third defining feature lies in how the BDF worksheets scaffold differentiation. The worksheets are structured to guide students from visual-concrete tasks toward increasingly symbolic tasks, ensuring that learners with lower prior knowledge can rely on visual cues while higher-achieving students can extend their strategies and articulate generalizations. This scaffolded approach addresses the diverse learning needs within a classroom, allowing for personalized learning trajectories while maintaining a coherent instructional framework

(Merkel, et al., 2025). This tiered design mirrors principles of guided reinvention (Gravemeijer, 1994; 1999) and supports adaptive learning, allowing students to progress through levels of abstraction without losing conceptual coherence.

The BDF manipulative demonstrated substantial benefits in the field test, yielding high performance scores of 91% - 99% across Tasks 1-3 in both VC and FS stages. These results indicate that the manipulative effectively fostered students' conceptual understanding for basic fraction concepts, equivalent fractions, and addition of fractions with the same denominator (Reinhold et al., 2020). The manipulative also supported students' transition into more complex operations, such as addition of fractions with unlike denominators, which resulted in an 88% score in the VS stage. This high level of performance across various fraction operations underscores the BDF manipulative's effectiveness in developing both conceptual understanding and procedural fluency, suggesting a strong alignment with findings that physical manipulatives enhance mathematical comprehension (Amo-Asante & Bonyah, 2024; Siller & Ahmad, 2024). This performance highlights the manipulative's role in scaffolding students' progression from concrete to abstract reasoning (Lemonidis & Pilianidis, 2020; Rich, 2023). However, the marked decrease in the FS score for Task 4 suggests that while the BDF manipulative effectively supports initial conceptualization in a visual–concrete context, students still face challenges when transitioning to purely symbolic manipulation, an issue particularly pronounced in non-intuitive operations such as fraction addition with unlike denominators (Amo-Asante & Bonyah, 2023; Pearn et al., 2022).

This finding reinforces the well-documented difficulty students experience when shifting from concrete, visual understanding to abstract symbolic reasoning in mathematics. It is consistent with Bruner's theory of representational development, which posits that understanding progresses from enactive (hands-on manipulation) to iconic (visual imagery) and to symbolic notation. The results also align with Tall and Vinner's distinction between concept image and concept definition, emphasizing that students' evolving concept images must be systematically connected to formal symbolic structures for deeper understanding to emerge (Tall & Vinner, 1981). The observed gap underscores the need for instructional designs that intentionally bridge visual–concrete and symbolic representations, potentially through adaptive scaffolding that gradually reduces reliance on visual support as students learn to translate seamlessly between concrete manipulations and abstract notation.

## Conclusion

This study highlights the role of the BDF manipulative in building students' conceptual understanding of fractions by providing a strong visual-concrete foundation and supporting coordinated visual-symbolic reasoning through integrated dual representations. The findings indicate that the manipulative can meaningfully support the development of fraction concepts. The results imply that the use of BDF manipulative together with students worksheets can help students build their understanding through Bruner's enactive-iconic-symbolic progression, thereby strengthening the connection between concrete actions and formal fraction notation. However, the observed performance gap in solving symbolic problems, particularly in the addition of fractions with unlike denominators, indicates that additional instructional guidance is needed to support students in transitioning from concrete representations to abstract procedures.

This study is limited by its specific sample context, which may affect the generalizability of the findings to broader educational settings. The relatively short duration of the intervention may also not fully capture longer-term developments in students' conceptual understanding. Future studies could examine the long-term effects of the BDF manipulative across more diverse student populations and investigate how combining dynamic digital tools with the BDF during early exploratory stages may further enhance students' cognitive engagement. Other studies might also explore instructional strategies that more effectively support the transition from visual-concrete reasoning to abstract-symbolic problem solving.

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## Declarations

| | | |
|---|---|---|
| Conflicts of Interest | : | The authors declare no conflict of interest. |
| Generative AI Statement | : | AI Used for Limited, Non-Substantive Support:<br>Scholar Labs was used to identify related studies and to assist the authors in preliminary brainstorming of potential research gaps.<br>Grammarly was employed solely for language editing and minor phrasing enhancements.<br>All conceptualization, methodological decisions, data interpretation, and substantive scholarly contributions were independently developed, validated, and finalized by the authors. |
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| Author Contributions | : | **Darmawijoyo:** Conceptualization of the idea and discussion; **Septy Sari Yukans:** writing original draft, editing, formal analysis, visualization; **Elika Kurniadi:** Data collection, data analysis; **Weni Dwi Pratiwi:** Research instruments, draft article review. |

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