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Students' commognition in solving linear programming question

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Abstract

Although mathematical communication is considered important, students still struggle to express mathematical ideas effectively. Therefore, this research is going to use a commognitive framework that provides a valuable lens for analyzing and enhancing students' mathematical communication skills. The purpose of this study is to describe students' cognition in solving linear programming problems. The researcher selected six students for interviews based on the consistency of their answers and then selected two students from the six students who had been interviewed. Commognitive analysis shows striking differences in mathematical thinking and communication between DAJ and ED subjects, who are students with high and low commognitive abilities, respectively. Students with high commognitive abilities tend to be more comprehensive and exploratory. In contrast, a student with low commognitive ability is relatively more limited and procedural. This implies that teachers cannot judge understanding only by whether students reach the correct answer. They must also attend to how students talk, write, and represent mathematics, since these discursive moves reveal whether their routines are genuinely conceptual or merely imitative. As a result, it is advised that future studies include a group discussion, better-developed question types, and more specified student criteria.

Keywords: cognition; commognition; communication; linear programming

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Introduction

Communication is a crucial aspect of mathematics in general and it also cannot be separated from mathematics in schools or any academic environment. Communication can turn an idea put forward by someone into something that can be used for self-reflection, discussion, and also improve one's abilities through the help of others (Habibi et al., 2023). The communication method also aids with comprehension. By asking students to do some reasoning through math problems and communicate their thoughts out loud or on paper, they also learn to explain their logic and influence others' understanding. (NCTM, 2000). Brodie (2010) agrees that mathematical knowledge is acquired through mathematical communication, arguing that communication is an essential component of understanding. Brodie (2010) also said that because communication is a vital aspect of knowing, students use it to share their knowledge with others.

According to the *Ontario Ministry of Education*, communication can be defined by active process of exchanging information and ideas to express thoughts that involves understanding and expression (Rohid et al., 2019). Students communicate for a variety of objectives and audiences, including professors, classmates, groups of students, and the entire class (Zayyadi et al, 2020). Communication plays a vital role in learning mathematics because it helps students reflect on ideas and deepen mathematical understanding (Biber, 2023; Tabacaru, 2016).

One of important aspect in mathematics education is mathematical communication (Daher, 2012; Kieran, 2001; Kosko, 2014; Lestari et al., 2019; Thinwiangthong et al., 2012). According to Sfard, mathematical communication is the process by which individuals transmit mathematical ideas in writing and vocally (Sfard, 2001, 2007, 2015). This skill is considered important in mathematics because it allows students to understand mathematical concepts on their own or make others understand them in written or oral form. Mathematical communication involves transmitting one's thoughts and arguments through mathematical representations and language, in both spoken and written (Susilowati et al., 2025).

Although mathematical communication is considered important, students' ability to express mathematical ideas effectively, especially in written form, is still not good (Pinto & Cooper, 2023; Rahmawati et al., 2023). This is even more evident when students are given problems that are different from those exemplified by the teacher, often causing them confusion when solving the problems. In addition, this is also influenced by a learning model that is too teacher-centered. Mathematical communication is more than just answering questions, it also includes reading, representing, and engaging in mathematical problems that are meaningful to students (Tong, Uyen, & Quoc, 2021). This process also includes expressing mathematical reasoning and creating logical connections between concepts (Kristiani et al., 2024). Based on the current educational framework, improving communication skills is very important in achieving learning objectives (Rahmawati & Soekarta, 2021). The importance of mathematical communication is not only focused on the discipline itself but also contributes to students' cognitive development and fosters deeper mathematical thinking.

When developing solutions to issues, a process of information processing happens. Information processing and individuals experience are cognitive process (Campos et al., 2013;

Iglesias-Sarmiento & Deaño, 2011; Sánchez et al., 2013; Montague et al., 2014). A person's cognitive processes include: (1) obtaining new information, (2) converting the information obtained, and (3) validate the truth of the information obtained (Sutarto, 2017). According to the previous statement, we can say that cognitive processes refer to how individuals acquire new information, store it in memory, and interpret it as a knowledge.

The cognitive process and effective mathematics learning of students can't be separated from the role of mathematical communication in it. Activities in which students sharpen their mathematical thinking through writing, speaking, or creating visual representations are examples of the cognitive process of mathematical information that impacts their overall mathematical abilities (Sukasno et al., 2024). The National Council of Teachers of Mathematics (NCTM) explains that communication is not only part of mathematics learning but also a basic skill needed to better understand mathematics (Shinno & Fujita, 2021). Communication skills have also been established as central to student mathematics development by the current modern education framework. Nauli et al. (2024) also emphasize that expressing ideas, reasoning, and demonstrating strategic competence in understanding mathematical concepts can improve the effectiveness of communication. Therefore, guiding the improvement of students' mathematical communication to be more organized, logical, and persuasive in both written and oral forms is very important because it prioritizes reasoning, accuracy, and argumentation as the core of mathematical literacy.

The interaction between communication skills and cognitive processes in mathematics education exerts a significant and multifaceted influence. Because of that, Sfard (2008) explain her work about commognitive framework that provides a valuable lens for analyzing and enhancing students' mathematical communication skills (Ching et al., 2020). By adopting strategies that prioritize precise word use, visual mediators, narratives, and routines, educators can substantially enhance students' capacity to express and comprehend mathematical ideas (Makgakga, 2023). This approach not only fosters individual cognitive development but also cultivates a collaborative and engaging learning environment, which is essential for students' academic and professional preparedness in mathematics (Chan et al., 2022; Matthews, 2024).

The portmanteau of communication and cognition is known as commognition (Sfard, 2001, 2006, 2008; Caspi & Sfard, 2012; Viirman, 2015; Kim et al., 2017). The mental processes and sharing of information with others or ourselves by verbal or non-verbal can be called commognition (Tasara, 2017). Commognitive research on high school students has been conducted by several scholars, including Akçakoca et al. (2024); Pratiwi et al. (2025); Roberts & Roux (2019); Wong et al. (2025); Zayyadi et al. (2019); Zulfah et al. (2025). Roberts & Roux (2019) analyzed junior high school students' discourse using the cognitive theory developed by Sfard (2008) to describe whether students used mathematically endorsable narratives to explain and justify their solutions. Pratiwi et al. (2025) studied students' mathematical communication and investigated how it reflects their algebraic thinking using commognitive-based learning that has been developed based on the commognitive framework. It also described the relationship between algebraic thinking and students' verbal communication skills statistically.

A qualitative case study conducted by Akçakoca et al. (2024) reported on whether high school students' mathematical insights regarding polynomial inequalities were primarily ritualistic or exploratory in nature. The students in the study worked on two polynomial inequality problems and by using Sfard's (2008) commognitive framework, the researchers examined student practices. In general, this study shows that student practices fall between ritual and exploration routines and the study concludes that teaching methods, learning activities, also task design can significantly help the transition to more exploratory mathematical discussions in solving inequality problems or even broader mathematical material.

Wong et al. (2025) conducted an exploratory qualitative study using a commognitive framework invented by Sfard (2008) to analyze how students use mathematics when collaboratively solving quantitative science problems (especially chemistry). Wong et al. (2025) concludes that the commognitive lens is a powerful tool for understanding how students combine mathematics and science in problem solving, and for designing instruction that makes these discourse patterns visible and productive. Additionally, the authors note that students' commognitive abilities can be evaluated through their written mathematical work, which reveals patterns in word use, visual mediators, narratives, and routines. Collectively, these studies demonstrate that commognition offers a comprehensive framework for understanding how students think, communicate, and learn mathematics in both problem-solving and broader instructional contexts.

From the previous study we saw that none of them talk about one of the most difficult topics in mathematics for students is solving application problems, such as those found in linear programming. This topic involves three equations consisting of two variables, with one equation acting as the objective function and the others as constraint functions, requiring students to understand symbols, graphs, and procedural understanding. Linear programming has numerous applications in various fields. However, many students still struggle to understand the concept because it's not very relevant, which can lead to a lack of motivation to learn it (Ho et al., 2019; Mwambazi, Mbewe, & Simui, 2025).

To resolve these challenges, the commognitive framework offers a meaningful theoretical approach by integrating the cognitive and communicative aspects of mathematical thinking (Lu et al., 2022). This framework has four main components, those are: (1) word use, (2) visual mediators, (3) narratives, (4) and routines (Chan et al., 2022; Matthews, 2024). Word use refers to students' ability to express known and unknown elements when using precise mathematical language. Visual mediators include the use of graphs, tables, and diagrams to represent mathematical relationships. Narratives involve structured explanations of mathematical facts, such as theorems or properties, while routines represent problem-solving procedures and steps used to reach a solution. As supported by Zayyadi et al. (2019), the application of the commognitive framework can improve students' ability to solve mathematical problems.

Research on commognition related to linear programming special cases with multiple solutions has not yet been conducted by other researchers. Therefore, it is necessary to conduct a study on student commognition using linear programming material with non-routine question types in order to describe student commognition in solving special cases of linear programming questions.

Methods

The type of research used was descriptive qualitative research. This type of research was chosen because the general purpose of qualitative research is to explain a situation or phenomenon that occurs to obtain qualitative data (Creswell, 2009). This aligns with the objective of this study, which is to describe students' commognition in solving linear programming problems. This study consists of four stages, namely: (1) preparation, (2) research subjects and location, (3) data collection, and (4) data analysis.

During the preparation stage, researchers developed test questions and interview guidelines that could help researchers analyze students' written answers in solving linear programming problems in greater detail. The test instruments include mathematical questions shown in Figure 1 that can reveal students' cognition in solving linear programming questions. The interview guidelines have also been designed in accordance with Table 1 to be in line with students' thinking processes when they solve linear programming questions. The following are the linear programming questions given to students:

> Mr. Joko has a business that focuses on the production of necklaces and bracelets. Mr. Joko uses a machine that can produce necklaces in 3 minutes and bracelets in 2 minutes. However, because the machine is quite old, it can only be operated for a maximum of 32 minutes at a time before it needs to be rested. The packaging area required for necklaces and bracelets is 50 cm^2 and 60 cm^2, respectively, with a maximum total packaging area of 720 cm^2. If the profit per necklace is Rp 18,000 and the profit per bracelet is Rp 12,000, how many of each product must Mr. Joko produce to maximize his profit?

Figure 1. Linear programming question

In the concept of linear programming, this problem is rarely encountered by high school students because one of its solutions is not an integer, and there are multiple solutions to the problem because one of the constraint functions is parallel or coincides with the objective function. By using problems such as this, it is hoped that students can show their commognition components in accordance with the indicators in Table 1.

Table 1. Commognition indicators used in solving linear programming question

Commognition Component		Indicator
Word Use	Literate	Using mathematical keyword to solve the question
	Colloquial	Using combination of mathematical and colloquial keyword to solve the question
Visual Mediator	Symbolic Mediator	Using mathematical representations equations, formulas, and mathematical notation to solve problems.
	Iconic Mediator	Using visual representations in the form of images such as tables, graphs, diagrams, and sketches.
Narrative	Remember and Explain	Capable of elucidating rationale and relating items, connections with earlier content, and procedures, including definitions, theorems, and proofs in problem-solving.
Routine	Ritualized	Answer the questions using the previously described method.

Commognition Component	Indicator
Exploratory	Resolving issues outside of the previously described methods and connecting them to ideas outside of the mentioned content.

(Adapted from Sfard, 2008; Mpofu & Pournara, 2018)

This study was conducted at one of the schools in Sidoarjo on 35 students in the 11th-grade odd semester. The students in this class are the top students in the school and have already learned linear programming in the 10th grade. The researcher choose the 11th-grade because the 10th-grade students at this school have not yet learned linear programming. The subjects in this study went through two stages, namely a written test and an interview. In the written test stage, the researcher selected six students for interviews based on the consistency of their answers. Then, after the interview stage was completed, the researcher selected two students from the six students who had been interviewed using the extreme sampling technique where the researcher selected 2 students called student with high & low commognition because the results of the commognition component analysis of the other 4 students were in the middle of the 2 selected students.

The data collection process in this study began with a written test about linear programming math questions that had been compiled based on students' commognition indicators and validated by experts to accurately measure students' commognition. Next, the researcher conducted semi-structured interviews to deepen and clarify the information obtained from the students' test results.

During the analysis stage, researchers obtained data on students who were asked to express their thoughts verbally through questions given during interviews. The researcher gave students the freedom to look up any math terms they might have forgotten to help them explain their explanation more clearly and structured during the interviews. The researcher also ensured that they only looked up math terms, not answers to question, during the interviews. During the interviews, the researchers also observed and recorded audio conversations that took place while the students reread their work. From this data, the researchers condensed the data by sorting out which data was deemed appropriate and relevant to the research conducted by the researchers and worthy of in-depth analysis.

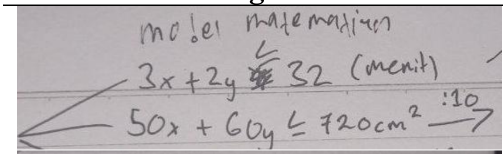
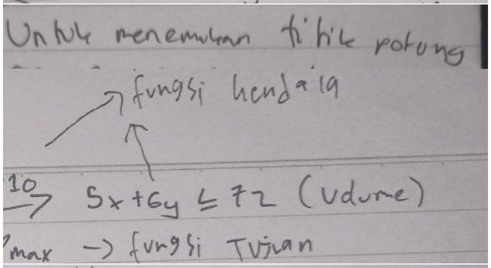
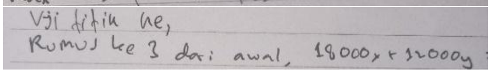
Results

In this section, data and analysis of the problem formulation are presented. This study identifies key findings regarding commognition experienced by students in solving problems on linear programming material: (1) word use (mathematical sentences used), (2) visual mediators (illustrations made by students), (3) narratives (theorems, definitions, axioms, and mathematical concepts used), and (4) routines (how students use narratives logically). The following will explain the findings related to commognition experienced by students.

Student with high commognition

First, researchers will analyze students' word use in solving linear programming problems given to DAJ. The following are DAJ's answers in terms of word use, as shown in Table 2.

Table 2. Results of DAJ questions in word use component

Figure	Translation	Mathematical Terms
	Mathematical model $3x+2y \leq 32$ (minutes) $50x+60y \leq 720cm^2$	Mathematical Model
	To find the intersection Constraint Function $5x+6y \leq 72$ (should be) Max→Objective function	Intersection Objective and Constraint Function
	Point test into, The third formula from beginning, 18,000x+12,000y	Point Test

In the word use component, DAJ students wrote down several mathematical terms such as "mathematical model" and the steps they took to solve the given problems. However, the students did not write down the examples represented by the variables x and y. DAJ also wrote down the word "intersection" to find the coordinates of the corner point of the solution set. Other mathematical terms written down were "objective function" and "constraint function," which are used to find the maximum profit based on the known number of necklaces and bracelets. In addition, DAJ also wrote down the word "test point" to find out which point would produce the maximum profit and finally, DAJ wrote a conclusion to explain how many items could produce the maximum profit. In this conclusion section, DAJ also wrote the mathematical phrase "maximum profit".

Since DAJ did not write down the examples, the researcher confirmed them during the interview stage with the conversation excerpts shown in Table 3.

Table 3. Transcript of interview with DAJ on word use component

R/DAJ	Transcript of Interview
R	*"When working on this problem, you have created a mathematical model in the form of an inequality. How did you obtain this form?"*
DAJ	*"I am using an analogy, sir, where the variable x represents the number of necklaces and y represents the number of bracelets."*
R	*"Then why does the inequality sign look like that?"*
DAJ	*"The inequality is less than or equal to because the question contains the words 'at most' and 'maximum'. That is why I am sure that the value must be less than or equal to, so I used that sign."*

The interview transcript in Table 2 shows that DAJ has a good understanding of what is known in the question and how to represent the story problem in mathematical form, although there are still some important aspects that could have been written down but were omitted by

DAJ. Therefore, it can be concluded that DAJ's word use component was conveyed well, even though it was not complete in its written form.

In the next stage, researchers analyzed the visual mediator component based on the results of DAJ's written answers. The Cartesian graph created by DAJ is shown in Figure 2 below.

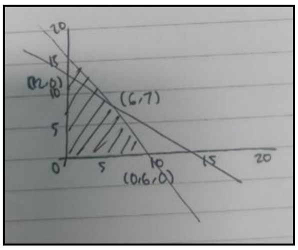


Figure 2. Results of DAJ visual mediator component question

Figure 2 shows that DAJ has successfully sketched a Cartesian graph of a mathematical model in the form of a 2 variable linear inequalities that they had previously determined. It can also be seen that the solution area is marked with black shading pointing to the origin (0,0). In DAJ's work above, it can also be seen that the students only shaded once and only wrote down three intersection points. In addition, DAJ wrote the coordinates of one of the points as "(0.6,0)" when DAJ's calculation clearly stated "(10.6,0)".

Due to several errors in the work performed by DAJ, the researcher confirmed this during the interview stage with the conversation excerpts shown in Table 4.

Table 4. Transcript of interview with DAJ on visual mediator components

R/DAJ	Transcript of Interview
R	*"When drawing a cartesian graph that illustrating the mathematical model you created, why did you only make one shading?"*
DAJ	*"I made one sketch there because if I made two sketches, it would be double, sir."*
R	*"Isn't it usually the case when explained by the teacher it becomes double like that?"*
DAJ	*"Yes, sir, but I prefer just one shade. It looks better that way."*
R	*"Usually, even if you use that method, won't you get criticized by the teacher?"*
DAJ	*"No, sir, because the result is still correct even though I only used 1 shading."*
R	*"Next, why did you only write down 3 of the 5 intersection points?"*
DAJ	*"Because I only focused on writing down the intersection points that serve as the corner points of the solution area, sir."*
R	*"But for this point, why did you write (0.6,0)?"*
DAJ	*"That's because the number 1 and the open parenthesis look almost the same, sir. So I thought I was just writing the number 1 without the open parenthesis."*

Based on the interview transcripts presented in Table 4, we know that students only made one shading because they wanted neater results. However, this caused students to make mistakes in shading the graph. In addition, the reason students only wrote three points because the students don't want too much writing on the graph. Finally, the students had a misperception in thinking that the closing parentheses were the number 1. Therefore, it can be concluded that the students' visual mediator component still had some errors caused by their understanding and how they solved the questions given by the researcher, which means that DAJ drew and explained the graph he made quite clearly even though there were a few errors.

In the third stage, researchers analyzed narrative components based on the DAJ responses shown in Figure 3.

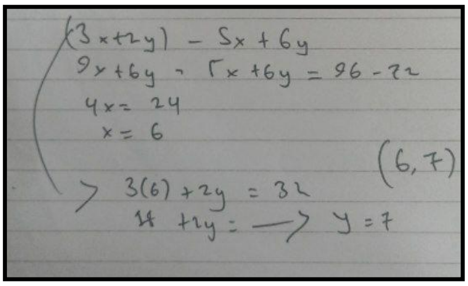


Figure 3. Results of DAJ component narrative questions

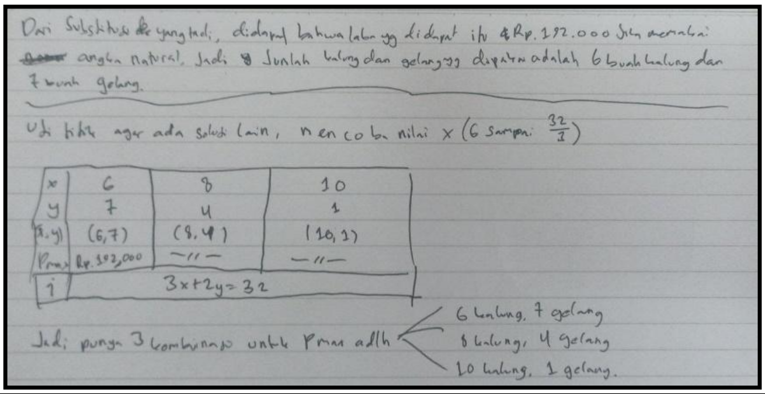
Figure 3 above shows how DAJ calculates the intersection between two constraint functions. However, the researcher was initially confused because this method is neither substitution nor elimination due to its significantly different procedure and form. Therefore, the researcher confirmed this in the interview stage with the interview transcript excerpts in Table 5.

Table 5. Transcript of interview with DAJ on the narrative component

R/DAJ	Transcript of Interview
R	*"Which method did you use to find the intersection point between graphs?"*
DAJ	*"I used the elimination method, Sir."*
R	*"But why does it look like you performed substitution?"*
DAJ	*"That is because the process is essentially the same if i write it like this, Sir. If i compute it using the conventional notation it would take up a lot of space."*
R	*"Then how in this step can the term '3x + 2y' become '9x + 6y'? "*
DAJ	*"That's because I multiplied everything by 3, Sir. Like in the usual elimination method."*
R	*"And here, why did '= 96 – 72' appear out of nowhere? Previously, that wasn't there"*
DAJ	*"Because from the original equation '3x + 2y = 32', I first wrote '3x + 2y' then multiplied by 3. The result of 32 times 3 is only written on the line below."*
R	*"And for this part, you wrote "–5x + 6y". Shouldn't that lead to "4x + 12y = 24"? Why is the "12y" missing? "*
DAJ	*"That's because I forgot to add parentheses. But I understood that the two variables on the left should be subtracted by the two variables on the right."*

Based on Table 5, it is known that the method used by students is the elimination method, but the calculation procedure used is written in accordance with DAJ's understanding so that DAJ can write down the calculations without taking up too much space. In addition, there are a few errors in the students' writing of calculation notation, which could cause them to make mistakes in their calculations. Therefore, it can be concluded that the DAJ Narrative component is quite good because DAJ is able to explain how he uses definitions, theorems, and existing procedures.

Then, in the final stage, researchers focused on analyzing the components of DAJ students' routines shown in Figure 4.



Translation :
From the substitution above, we obtain a profit of Rp 192,000 when using natural numbers. Therefore, the number of necklaces and bracelets produced is 6 necklaces and 7 bracelets.
Test the point to see if there are other solutions by trying values of x (6 to 32/3).

x	6	8	10
y	7	4	1
(x,y)	(6,7)	(8,4)	(10,1)
P_{max}	Rp 192,000	Rp 192,000	Rp 192,000
i	3x+2y=32		

So, the 3 combinations for P_{max} are 6 necklaces and 7 bracelets, 8 necklaces and 4 bracelets, and 10 necklaces and 1 bracelet.

Figure 4. DAJ routines component work results

Previously, the DAJ subject performed the linear programming problem-solving procedure as taught by his teacher. Starting from creating a mathematical model, making graphs, to determining the corner points. However, in the conclusion section, the DAJ subject still performed another procedure, namely trying several other values. This attracted the researcher's attention to find out more about the reasons why students performed this procedure through interviews, the transcripts of which are in Table 6.

Table 6. Transcript of interview with DAJ routines component

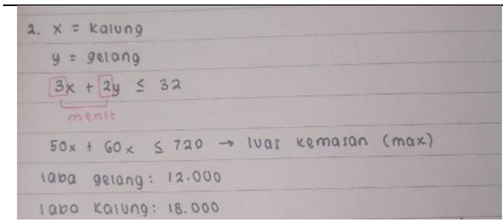
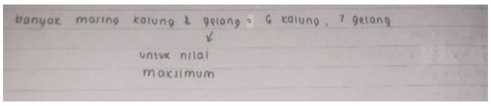
R/DAJ	Transcript of Interview
R	*"Why did you perform this procedure?"*
DAJ	*"I saw that the equation 3x+2y=32, the front part "3x + 2y" is a simplification of my objective function whose equation was 18x+12y."*
R	*"What happens if both equations are multiples of each other?"*
DAJ	*"As far as I remember, they are parallel lines, Sir."*
R	*"Then why, when they are parallel, did you still perform that procedure?"*
DAJ	*"I don't know, Sir; I was just doing it on a whim."*
R	*"Why did you choose the numbers 6, 8, and 10?"*
DAJ	*"I chose integer numbers from the graph I made, Sir. But I only wrote 6, 8, and 10 on the answer sheet because those were equal to the maximum profit value."*
R	*"Was this suggested by your teacher before?"*
DAJ	*"No, Sir."*
R	*"Do you think that answer can still be used?"*
DAJ	*"Yes, Sir."*

Table 6 shows that DAJ subject knew that the objective function had the same gradient as one of the existing constraint functions. That is why the DAJ subject tried several corresponding x and y values that had maximum benefits. In addition, DAJ subject also said that he performed the procedure outside of what his teacher had taught him. Therefore, it can be concluded that the DAJ routine component is a form of exploration because DAJ performed a procedure that was not previously explained by his teacher, even though DAJ was still not entirely sure how it could produce the same maximum profit.

Student with low commognition

Next, the researcher will analyze the word use of ED students in solving the given linear program questions. The following are ED's answers on the word use component as shown in Table 7.

Table 7. Results of ED questions in word use component

Figure	Translation	Mathematical Terms
	x=necklace y=bracelet $3x + 2y \leq 32$ Minutes →Packaging area (max) Bracelet Profit : 12,000 Necklace Profit : 18,000	Denotation of the Necklace and Bracelet
	Ammount of bracelet and necklace=6 necklace,7 bracelet For maximum value	Maximum Profit

In the word use component, ED students wrote down several mathematical terms such as "maximum profit" and the steps they took to find the value of the maximum profit. ED also wrote down variable examples to facilitate the calculation. However, the students did not write down the words "mathematical model", "objective function", "constraint function", "intersection point", and "point test." Finally, ED also wrote a conclusion explaining the quantity of goods that would yield maximum profit, even though the student did not explicitly mention the nominal profit obtained.

Since ED did not explicitly write down several important terms and nominal profits, the researcher confirmed them during the interview stage with conversation excerpts shown in Table 8.

Table 8. Transcript of interview with ED on word use component

R/ED	Transcript of Interview
R	*"When working on the word problem given, you transformed it from words into an inequality form. Do you know what that form of equation is called?"*
ED	*"It's called a mathematical model, Sir."*
R	*"Do you know what is meant by objective function and constraint function?"*
ED	*"As far as I know, the objective function is basically the final result we want to obtain (maximum profit), and the constraint function is the rule that limits so that the items are not too many or too few."*

R	*"Why didn't you write down the objective function equation in your answer?"*
ED	*"Because I was confused how to write it, Sir. Therefore I only wrote the profit per item."*
R	*"How did you know that during the point-testing (test point) you could substitute the values there?"*
ED	*"Because from the beginning I was asked to find the profit, Sir."*

The interview transcript in Table 8 shows that ED already has a good understanding of what is known in the question and how to represent the story problem in mathematical form, although there were still some important aspects that could be written down but were omitted by ED. When creating a mathematical model, ED used his own logic to classify which ones were constraint functions and which ones were functions. Therefore, it can be concluded that ED's word use component is well conveyed, although it is incomplete in its written form and tends to be less than DAJ's.

In the next stage, researchers could not analyze the visual mediator component based on ED's written answers because ED did not draw any graphs on the answer sheet. Since ED did not draw graphs to determine the solution set, researchers confirmed this in the interview stage with the conversation excerpts shown in Table 9.

Table 9. Transcript of interview with ED on visual mediator components

R/ED	Transcript of Interview
R	*"Why didn't you draw the graph of the solution set?"*
ED	*"Because I'm confused, Sir."*
R	*"Which part made you confused?"*
ED	*"I'm confused because I was only able to draw one line from the point I found. Meanwhile, my friend managed to get two lines."*
R	*"Was your working process and your friend's the same?"*
ED	*"Yes, Sir."*

The interview transcript in Table 8 shows that ED was confused when trying to make a graph of the solution set from the linear programming problem he was working on. ED argued that he only got two points which, when connected, would only produce one line, unlike his friend who was able to make two lines so that he could make the desired solution set graph. Ultimately, the researcher discovered a writing error made by ED, which will be discussed in the narrative component.

In the third stage, researchers focused on analyzing students' narrative components based on the results of their written ED answers, as shown in Figure 5.

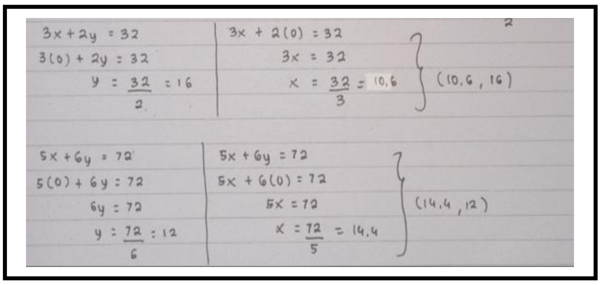


Figure 5. ED narrative component work result

Figure 5 shows how ED finds the intersection points of the two constraint functions on the x-axis and y-axis, as well as the intersection point between the two constraint functions. In the calculation results in the upper left corner, it can be seen that the student only wrote down the coordinates of one point, namely "(10.6,16)", instead of writing down the coordinates of two points, "(0,16)" and "(10.6,0)". The same applies to the calculation results in the lower left corner. Due to the unique results, the researcher confirmed the students' intention in writing these results during the interview stage with the transcript presented in Table 10.

Table 10. Transcript of interview with ED on narrative component

R/ED	Transcript of Interview
R	*"Why did you write the point as (10.6,16) only? Also for the one below?"*
ED	*"I wrote it like that because beside it I got that y = 16 and x = 10.6. Therefore I wrote it like that, Sir."*
R	*"But here you assumed x = 0 and here you assumed y = 0, shouldn't that result in two points?"*
ED	*"Oh, is that so, Sir? I don't know how to interpret that."*
R	*"That's okay. So if the result has two points, where are those points coordinate?"*
ED	*"They are at (0, 16) and (10.6, 0), Sir."*

Based on the interview transcript in Table 10, we can see that the reason ED only wrote down the points (10.6,16) was because ED assumed that the x and y values ED had found were the coordinates of the same point. This prevented ED from graphing the mathematical model ED had created on the Cartesian coordinate system.

Then, in the final stage, the researchers focused on analyzing the components of ED students' routines shown in Figure 6 below.

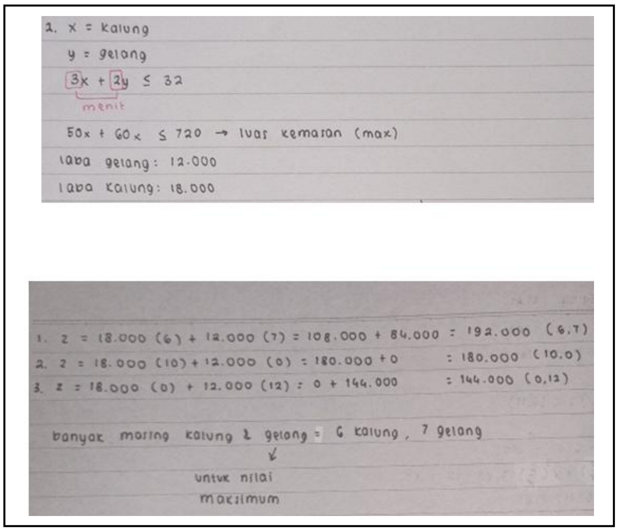


Figure 6. ED routines component work result

In Figure 6, it can be seen that the ED subject worked on the linear programming problem according to what was taught by the teacher. Unlike the DAJ student, who realized that there was a constraint function that had the same gradient as the objective function, the ED student did not realize this. Therefore, it can be concluded that the ED routine component was a form of ritual because ED only used the procedures that had been taught by the teacher previously.

Discussion

This study has explained students' commognition in solving linear programming problems, namely the mental process and communication of information to oneself or others in the process of expressing ideas to solve problems, which is done verbally and non-verbally, focusing on linear programming material. In this study, DAJ also represented students with high commognitive ability and ED with low commognitive ability. In terms of word use, high commognitive students exhibit more precise and comprehensive word use than low commognitive students. A high ability student in the study comfortably blended formal mathematical terms with everyday language by showing flexibility, strong grasp of concepts, and a deep understanding of the mathematics problem context. By contrast, the low-ability student used a much more limited repertoire of mathematical words, often omitting key terms such as objective function or constraint in a linear programming task, and relying on informal language. Their written solution lacked precision like they did not explicitly label the mathematical model or important quantities, reflecting an inconsistent use of vocabulary and an intuitive but informal grasp of concepts. In interviews, they could verbally identify some proper terms when guided, which reveals that his understanding was partially there but not manifested in written discourse. The result states that low-commognitive students may know basic terminology but they struggle to deploy it appropriately in problem-solving. Therefore, students with stronger mathematical commognitive ability demonstrate higher levels of understanding and success in problem solving (Khadka, 2024). In summary, high commognitive students integrate and use mathematical terminology more effectively, whereas low commognitive students often using imprecise word use that can hinder clear mathematical communication (Supardi et al., 2021).

In the study, the high commognitive student drawn a clear Cartesian graph of the linear programming model, marking the feasible region and intersection points of constraints which is an action demonstrating the ability to translate an abstract problem into a visual form. They employed both iconic and symbolic representations by drawing the constraint lines, shading the region, and using algebraic notation for coordinates, which illustrates a capacity to fluidly shift between the problem's visual and algebraic aspects. Meanwhile, the low-ability student did not produce a useful diagram and avoided drawing the graph of the solution set entirely, admitting confusion about how to plot the constraints when he found only one line where his peer found two. Prior studies have found that successful problem solvers frequently draw and refer to diagrams, and such behavior is linked to higher achievement on complex tasks (Heldeman et al., 2017). By not utilizing visual tools or by using them in solving the problem, low-commognitive students miss opportunities to conceptualize the problem spatially. Thus, high-commognitive students tend to leverage visual mediators more fully whereas low-commognitive students often underutilize these visual tools, which can impede their understanding of the problem's structure (Nardi et al., 2014; Ratumanan et al., 2022).

When constructing mathematical narratives, high-commognitive students produce more coherent and complete reasoning than low-commognitive students. The high commognitive ability student was able to articulate a clear narrative of their solution process. They reasoned logically through each step and even verbalized their thought process during the interview. This included correctly recalling relevant information and monitoring his progress (Valenta et al., 2024). In contrast, the low-ability students’ narrative had significant gaps and misunderstandings until the interviewer corrected it, indicating that low commognitive students’ narrative was not self-sufficient or fully internally consistent. These findings highlighted by Yerizon et al. (2025) that if the student tend to depens on procedural/algebraic routines rather than conceptual understanding there is a high risk that they will misunderstand their solutions, even if they follow standard procedures. Therefore, high-commognitive students formulate clearer and more logically narratives during problem-solving, while low-commognitive students frequently struggle to articulate a complete and correct reasoning process, sometimes stopping short or needing external guidance to fill in the gaps (Roberts & Roux, 2019).

This difference in understanding narratives is in line with the difference in routine components. High-commognitive students are more willing to depart from routine procedures and explore novel approaches when needed, whereas low-commognitive students stick closely to taught algorithms. In the study, high-commognitive students was able to recognize this special condition and respond by testing multiple feasible points, indicating the emergence of exploratory routines. This behavior suggests that high-commognitive students did not rely solely on previously learned algorithms but was able to adapt and extend his discourse when encountering a non-routine situation. The method used by students was not conveyed directly by the teacher but was done spontaneously by students due to special cases in the questions that caused students with high commognitives to perform exploratory routines. Conversely, students with low commognitives followed the steps in the book/taught too closely and did not realize the special cases in the questions categorized by Sfard (2008) as ritual routines. These findings are also in line with the statement by Roberts & Roux (2019) that many middle school students still use standard steps in solving problems involving equations and do not adapt their solution strategies when faced with new types of problems. In addition, low-commognitive students' routines are less flexible and do not produce all the optimal solutions available. Meanwhile, the high-commognitive students’ routine is quite flexible because it produces all the desired optimal solutions. In short, students with high cognition can modify the knowledge they have acquired to solve different problems. Meanwhile, students with low cognition are very dependent on the procedures that have been taught without any modification even when they encounter new problems (Akçakoca et al., 2024).

Conclusion

The written and verbal answer of students with high commognitive ability tend to be clearer, more detailed, coherent, and structured in their mathematical sentence construction, diagram creation, and consistent use of theorems. Presenting results in this way can enable students to easily solve non-routine problems and deepen their understanding of concepts. On the other hand, students with low commognitive ability did some incomplete, incoherent, and unstructured arguments when faced with new problems because they are too fixated on procedures. Which indicates that their mathematical cognition and communication are still not good enough. When working on fairly common problems, these two students do indeed have the same results. However, students with low commognitive ability are more prone to conceptual errors, misconceptions, and recognizing special cases. Therefore, teachers must implement various strategies that can support students in mastering linear programming material better and teachers cannot judge understanding only from whether students reach the correct answer. They must also attend to how students talk, write, and represent mathematics, since these discursive moves reveal whether their routines are genuinely conceptual or merely imitative. This problem arises as a result of the inability of students with low commognitive abilities to solve problems that are new to them.

To overcome the limitations of this study, it is recommended that future research consider the following. First, the use of group discussions among students to help broaden the scope of analysis and enable observation of changes in student discourse throughout the interaction. Second, the types of questions should be further developed, for example, by using a system of equations that produces corner points with non-integer coordinates but provides integer solutions at a position along the constraint line so that the test can challenge students' reasoning at a higher level of abstraction. Third, to prevent the results from being too broad, further studies should add specific selection criteria for the students who are the subjects, so that the characteristics of the respondents are more defined and specific.

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