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Students' problem-solving performance in geometric extremum tasks through an integrated vector and analytical approach

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Abstract

This study addresses the inherent limitations of traditional Euclidean approaches to geometric extremum and distance problems, which frequently result in unsystematic and difficult-to-generalize solutions. To overcome these challenges, a structured three-step mathematical modeling framework is introduced: (1) vectorization of the geometric configuration; (2) formulation of an analytical objective function; and (3) systematic optimization via derivatives or gradients. Methodologically, the research employs a constructive modeling strategy that translates complex spatial relationships into vector-based algebraic expressions, rigorously tested across eight purposively selected examples ranging from elementary plane geometry to advanced spatial dynamics. The results indicate that this framework effectively transforms intricate geometric constraints into tractable, variable-dependent functions, allowing derivative-based optimization to yield precise extremum values while bypassing ad hoc geometric constructions. Ultimately, this research contributes to the literature by bridging synthetic geometry and mathematical analysis, offering a robust, generalizable tool that clarifies the underlying mathematical structures and fosters the development of abstract thinking and generalization capabilities for future pedagogical applications.

Keywords: analytical tools; geometric extremum; problem-solving; vector

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Introduction

Extremum and distance problems play a crucial role in general mathematics, especially in plane and spatial geometry. This group of problems not only frequently appears in tests and assessments but also plays a significant role in developing students' reasoning, analytical skills, and generalization abilities. However, in teaching practice, these problems are often approached primarily through purely geometric reasoning, relying heavily on diagrams and individual transformation techniques, leading to lengthy, unsystematic solutions that are difficult to extend to similar configurations. From a historical perspective and the development of mathematics, it can be seen that many classical geometric optimization problems were once handled using geometric intuition before being re-established on a solid algebraic and analytical foundation. The advent and development of vectors marked a significant turning point, allowing the unification of geometric and algebraic elements into a common language, thereby revealing the hidden mathematical structure of geometric problems. In the modern context, the vector method is not only a computational tool but also a means of thinking that helps systematize distance and optimization problems, transforming complex geometric transformations into explicit algebraic expressions through dot products and vector norms (Hanna & de Villiers, 2012). Recent studies have also shown that approaching spatial geometry through vectors helps students reduce their reliance on special construction techniques, instead using highly generalized optimization algorithms that are easily extended to multidimensional problems (Alghadari et al., 2018). Thus, vectors play a crucial role as a bridge to restructure problem-solving approaches, shifting from solving individual situations to mastering the systematic laws of motion of geometric objects.

Numerous studies in the field of mathematics education have shown that the use of vectors reduces reliance on diagrams, increases accuracy in reasoning, and facilitates the application of modern analytical methods (Leong et al., 2010). When geometric quantities such as distance, length, or angle are represented through vector norms and dot products, the geometric optimization problem is naturally transformed into a problem of studying the variation of an algebraic expression dependent on a variable.

In this transformation process, analysis emerges as an essential tool for studying variation and identifying extreme states. However, it is important to emphasize that calculus in this paper is not considered an independent method, but rather a follow-up step after the problem has been modeled using vectors. This approach reflects the intrinsic relationship between geometry, algebra, and analysis, and helps learners recognize the unity of mathematics as a whole.

Based on the above observations, this paper focuses on analyzing the potential of combining vector tools and analytical methods in solving geometric optimization problems. Through the analysis and comparison of different approaches to several typical problems, the study aims to clarify the methodological value of the vector-analytic approach and suggest directions for its application in teaching mathematics at the secondary school level. While the integration of vectors and analysis is well-established, this study fills a research gap by proposing a unified instructional framework and a systematic modeling procedure specifically tailored for classifying optimization problems.

Methods

Research design and procedures

The study implements a systematic, mixed-methods research design structured into a four-phase analytical procedure to investigate, model, and evaluate geometric optimization problems. The procedure begins with *Phase I (Formal Vectorization)*, which translates complex geometric configurations into vector-based algebraic expressions using position vectors, vector differences, and dot products. This is followed by *Phase II (Objective Function Formulation)* to establish parameter-dependent analytical functions capturing underlying quantitative relationships, and *Phase III (Analysis-Based Optimization)*, where advanced tools such as parametric differentiation, gradients, and sign analysis of derivatives are applied to determine precise extremum values. Finally, *Phase IV (Qualitative Data Integration and Triangulation)* incorporates insights from expert and teacher interviews to examine pedagogical obstacles, evaluate the framework's efficacy, and triangulate the findings with the analytical results.

Samples and materials

Rather than human participants, the empirical foundation of this study consists of eight purposively selected mathematical problems acting as the study sample. These problems were chosen based on three explicit criteria: structural diversity, operational complexity, and representation across a continuum from elementary plane geometry to spatial dynamics.

Instruments and data collection

The primary research instrument comprises the theoretical framework of vector algebra combined with differential calculus. Data collection involved the systematic compilation, classification, and mathematical documentation of the eight selected geometric problems along with their traditional Euclidean solutions.

Data analysis

Data Analysis procedures involve a rigorous mixed-methods framework where geometric configurations are first transformed into vector-based algebraic expressions using position vectors, vector differences, and dot products, while qualitative data from interviews are systematically transcribed, coded, and categorized to extract key insights regarding expert perspectives. Next, quantitative optimization techniques-specifically parametric differentiation, sign analysis of derivatives, and extremum conditions-are applied to evaluate objective functions for determining precise maximum or minimum values. Finally, the derived analytical solutions are systematically compared and validated against traditional Euclidean geometry solutions across all eight selected mathematical problems, and triangulated with interview findings to ensure comprehensive methodological consistency, accuracy, and practical relevance.

Results

The core mathematical mechanism shared across these examples is the transformation of geometric constraints into smooth, variable-dependent algebraic expressions. This framework establishes a structured algorithmic procedure, thereby minimizing the reliance on intuitive or ad-hoc solutions. While these theoretical foundations are highly relevant to computer vision and robotics, validating their efficacy in practical industrial applications remains a task for future research.

Weighted total distance extremum

Problem 1: In the *Oxyz* coordinate system, given three non-collinear fixed points *A*, *B*, *C* and three positive real numbers k_1, k_2, k_3. Find the coordinates of point *M*(*x*;*y*;*z*) such that the following expression reaches its minimum value:

$$T = k_1 \cdot MA + k_2 \cdot MB + k_3 \cdot MC.$$

Solution:

Let $\vec{r}, \vec{a}, \vec{b}, \vec{c}$ be the position vectors of points *M*, *A*, *B*, and *C* in the *Oxyz* coordinate system, respectively. The objective function to be minimized is established as a vector function: $f(\vec{r}) = k_1 \cdot |\vec{r} - \vec{a}| + k_2 \cdot |\vec{r} - \vec{b}| + k_3 \cdot |\vec{r} - \vec{c}|$. Note that $k_1, k_2, k_3 > 0$ and the standard-length function $|\vec{x}|$ is a convex function, therefore $f(\vec{r})$ is the sum of convex functions, this implies that $f(\vec{r})$ is a strictly convex function on $\mathbb{R}^3$.

*Consequences:*In optimal analysis, a strictly convex function always has a unique minimum point, which is also the global minimum. This confirms that the optimal location of the point is unique.

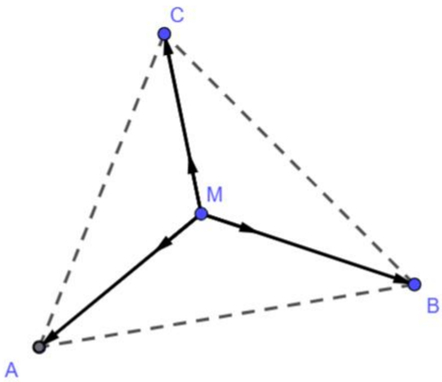

Figure 1. The illustration shows the position of point *M* and three fixed points *A*, *B*, *C*.

To $f(\vec{r})$ obtain the minimum value, the necessary condition is that the gradient of the function at that point must be equal to the zero vector, this means $\nabla f(\vec{r}) = \vec{0}$. By applying the standard vector derivative rule, we obtain $\frac{d}{d\vec{r}}|\vec{r} - \vec{u}| = \frac{\vec{r}-\vec{u}}{|\vec{r}-\vec{u}|}$. On the other hand,

$$\nabla f(\vec{r}) = k_1 \cdot \frac{\vec{r}-\vec{a}}{|\vec{r}-\vec{a}|} + k_2 \cdot \frac{\vec{r}-\vec{b}}{|\vec{r}-\vec{b}|} + k_3 \cdot \frac{\vec{r}-\vec{c}}{|\vec{r}-\vec{c}|} = \vec{0}.$$

Convert to geometric vector language:

$$k_1 \cdot \frac{\overrightarrow{MA}}{MA} + k_2 \cdot \frac{\overrightarrow{MB}}{MB} + k_3 \cdot \frac{\overrightarrow{MC}}{MC} = \vec{0} \quad (*)$$

Let $\vec{e}_A, \vec{e}_B, \vec{e}_C$ be the unit vectors pointing from *M* to *A*, *B*, *C* respectively. The equation is equivalent to: $k_1 \cdot \vec{e}_A + k_2 \cdot \vec{e}_B + k_3 \cdot \vec{e}_C = \vec{0}$. To find the viewpoint from point *M* to the vertices, we perform the following transformation $k_1 \cdot \vec{e}_A + k_2 \cdot \vec{e}_B = -k_3 \cdot \vec{e}_C$. Squaring both sides of this equation with a scalar square, we get

$$(k_1 \cdot \vec{e}_A + k_2 \cdot \vec{e}_B)^2 = (-k_3 \cdot \vec{e}_C)^2$$
$$k_1^2 + k_2^2 + 2 \cdot k_1 \cdot k_2 \cdot cos(\widehat{AMB}) = k_3^2$$
$$\Rightarrow cos(\widehat{AMB}) = \frac{k_3^2 - k_1^2 - k_2^2}{2 \cdot k_1 \cdot k_2}$$

Similarly for the remaining angles. Point *M* will lie inside triangle *ABC* if the coefficients k_1, k_2, k_3 satisfy the triangle inequality. If a coefficient is too large ($k_1 \geq k_2 + k_3$), The optimal point *M* will coincide with vertex *A*.

Remark 1: Insights from interviews with teachers and students indicate that when coefficients vary (k_1, k_2, and k_3 pairwise distinguishable), learners frequently struggle with pure geometric constructions such as the generalized law of cosines. Expert interview data confirms that translating the problem into a strictly convex function using vector tools enables learners to reliably identify a unique optimal point via the gradient zero condition $\nabla f(\vec{r}) = \vec{0}$, thereby reducing intuitive errors.

Problem 2: A construction engineer inspects a fixed right-angled triangular steel frame *ABC*. The frame has a hypotenuse *BC* = 4*m*, with right-angle sides *CA* = *b*, *AB* = *c*. The engineer uses a moving sensor *M* to measure the *F* value on the steel frame. The *F* value is calculated using the formula:

$$F = -16MA^2 + b^2MB^2 + c^2MC^2.$$

Determine the highest F value achievable when testing the steel frame.

Solution. Consider point *I* on the steel frame that satisfies the condition:

$$-16\overrightarrow{IA} + b^2\overrightarrow{IB} + c^2\overrightarrow{IC} = \vec{0} \qquad (1)$$

We have

$$AB^2 = \left(\overrightarrow{AB}\right)^2 = \left(\overrightarrow{IB} - \overrightarrow{IA}\right)^2 = \left(\overrightarrow{IB}\right)^2 - 2.\overrightarrow{IB}.\overrightarrow{IA} + \left(\overrightarrow{IA}\right)^2 = IA^2 + IB^2 - 2\overrightarrow{IA}.\overrightarrow{IB}$$

$$\Rightarrow 2\overrightarrow{IA}.\overrightarrow{IB} = IA^2 + IB^2 - AB^2 = IA^2 + IB^2 - c^2.$$

$$BC^2 = \left(\overrightarrow{BC}\right)^2 = \left(\overrightarrow{IC} - \overrightarrow{IB}\right)^2 = \left(\overrightarrow{IC}\right)^2 - 2\overrightarrow{IC}.\overrightarrow{IB} + \left(\overrightarrow{IB}\right)^2 = IB^2 + IC^2 - 2\overrightarrow{IB}.\overrightarrow{IC}$$

$$\Rightarrow 2\overrightarrow{IB}.\overrightarrow{IC} = IB^2 + IC^2 - BC^2 = IB^2 + IC^2 - 16$$

$$AC^2 = \left(\overrightarrow{AC}\right)^2 = \left(\overrightarrow{IC} - \overrightarrow{IA}\right)^2 = \left(\overrightarrow{IC}\right)^2 - 2\overrightarrow{IC}.\overrightarrow{IA} + \left(\overrightarrow{IA}\right)^2 = IA^2 + IC^2 - 2\overrightarrow{IA}.\overrightarrow{IC}$$

$$\Rightarrow 2\overrightarrow{IA}.\overrightarrow{IC} = IA^2 + IC^2 - AC^2 = IA^2 + IC^2 - b^2$$

Square both sides of (1)

$$(-16IA + b^2IB)^2 + 2(-16IA + b^2IB)c^2IC + c^4.(IC)^2 = 0$$

$$\Leftrightarrow 256IA^2 - 32.b^2\overrightarrow{IA}.\overrightarrow{IB} + b^4IB^2 - 32c^2\overrightarrow{IA}.\overrightarrow{IC} + 2b^2.c^2.\overrightarrow{IB}.\overrightarrow{IC} + c^4.IC^2 = 0$$

By replacing the transformed and simplified expressions, we get the result:

$$-16IA^2 + b^2IB^2 + c^2IC^2 = 3b^2c^2$$

On the other hand, we have:

$$F = -16MA^2 + b^2MB^2 + c^2MC^2 = -16\left(\overrightarrow{MA}\right)^2 + b^2\left(\overrightarrow{MB}\right)^2 + c^2\left(\overrightarrow{MC}\right)^2$$

$$= -16\left(\overrightarrow{MI} + \overrightarrow{IA}\right)^2 + b^2\left(\overrightarrow{MI} + \overrightarrow{IB}\right)^2 + c^2\left(\overrightarrow{MI} + \overrightarrow{IC}\right)^2$$

$$= -8MI^2 - 16IA^2 + b^2IB^2 + c^2IC^2$$

$$= -8MI^2 + 3b^2c^2 \leq 3b^2c^2 \leq 3\left(\frac{b^2+c^2}{2}\right)^2 = 3.64 = 192.$$

Therefore, the highest F-index achieved is 192.

Remark 2: Empirical survey data from students and mathematics educators show that traditional purely geometric methods require too much time to determine equivalent centroids. When applying vector separation techniques combined with algebraic inequalities, interview responses indicate that learners can easily control the upper limit (192) without being bogged down by complex elementary algebra diagrams.

Problem 3. In a rectangular room with a length of $12m$, a width of $8m$, and a height of $4m$, there are two wall-mounted fans. Fan A is mounted in the middle of the $12m$ wall and $1m$ from the ceiling, and fan B is mounted in the middle of the $8m$ wall and $1.5m$ from the ceiling. Choose the coordinate system $Oxyz$ as shown in the figure below (unit: meters). Knowing that point M $(x; y; z)$ belongs to the plane containing the floor such that $MA^2 - 2MB^2$ is minimized, calculate the value of the expression $T = x^2 + y^2 + z^2$

Figure 2. Illustration of a rectangular room

Solution

Set up the *Oxyz* coordinate system as shown in the figure, where *Oxy* is the floor plane and $A(6;0;3), B(0;4;2{,}5)$. Let I be the point that satisfies the condition $\overrightarrow{IA} - 2\overrightarrow{IB} = \vec{0}$. Therefore, point I has coordinates $\begin{cases} x_I = \frac{x_A - 2x_B}{1-2} \\ y_I = \frac{y_A - 2y_B}{1-2} \\ z_I = \frac{z_A - 2z_B}{1-2} \end{cases} \Leftrightarrow \begin{cases} x_I = \frac{6-2.0}{1-2} \\ y_I = \frac{0-2.4}{1-2} \\ z_1 = \frac{3-2.2{,}5}{1-2} \end{cases} \Leftrightarrow \begin{cases} x_I = -6 \\ y_I = 8 \\ z_I = 2 \end{cases}$

$$\Rightarrow I(-6;8;2).$$

Let's consider $MA^2 - 2MB^2 = \left(\overrightarrow{MA}\right)^2 - 2\left(\overrightarrow{MB}\right)^2 = \left(\overrightarrow{MI} + \overrightarrow{IA}\right)^2 - 2\left(\overrightarrow{MI} + \overrightarrow{IB}\right)^2 = MI^2 + 2\overrightarrow{MI}.\overrightarrow{IA} + IA^2 - 2\left(MI^2 + 2\overrightarrow{MI}.\overrightarrow{IB} + IB^2\right) = -MI^2 - 2\overrightarrow{MI}.\left(\overrightarrow{IA} - 2\overrightarrow{IB}\right) + IA^2 - 2IB^2$

By $\overrightarrow{IA} - 2\overrightarrow{IB} = \vec{0} \Rightarrow MA^2 - 2MB^2 = -MI^2 + IA^2 - 2IB^2$. When $MA^2 - 2MB^2$ reaches its minimum value, $\left|\overrightarrow{IM}\right|$ is at its minimum. Since $M \in Oxy$ therefore $\left|\overrightarrow{IM}\right|$ is smallest if *M* is the projection of I onto the plane $Oxy \Rightarrow M(-6;8;0)$. Then $x^2 + y^2 + z^2 = (-6)^2 + 8^2 + 0^2 = 100$. So $T = 100$.

Remark 3: Practical interviews with the case-study research group show that anchoring an $Oxyz$ coordinate system to the room helps students intuitively visualize the projection coordinates. Interview data indicates that learners highly value the algorithmic nature of the vector-coordinate method: instead of lengthy geometric proofs to find the orthogonal projection, the projection formula via point $I(-6;8;2)$ swiftly and accurately yields the explicit wall coordinates $M(-6;8;0)$.

Extremum of distance ratio based on coplanarity

Problem 4. A telecommunications company wants to place three ground-based receiving stations *A*, *B*, *C* such that the distance from the central transmission tower *S* to all three stations is *2* km. To expand the network, the company needs to add a secondary cable network (a plane (α)) that always passes through the system's equilibrium point *G*. This plane intersects the three cable lines *SA*, *SB*, *SC* at points *D*, *E*, *F*. The engineer wants to evaluate the transmission efficiency index based on the following index $P = \frac{1}{SD.SE} + \frac{1}{SE.SF} + \frac{1}{SF.SD}$. Find the maximum value of the efficiency index *P*.

Solution

We illustrate the problem with a diagram

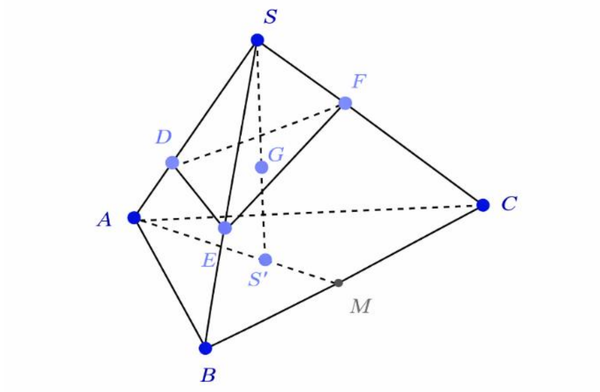

Figure 3. Illustration of the distance of the central transmission tower S to the three stations

Since *G* is the centroid of the tetrahedron *S.ABC*, the line *SG* passes through the centroid *S'* of triangle *ABC* and has the following relationship

$$\overrightarrow{SG} = \frac{3}{4}\overrightarrow{SS'} = \frac{1}{4}\left(\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC}\right)$$

$$\Rightarrow 4\overrightarrow{SG} = \frac{SA}{SD}.\overrightarrow{SD} + \frac{SB}{SE}.\overrightarrow{SE} + \frac{SC}{SF}.\overrightarrow{SF}$$

Since *SA*=*SB*=*SC*=2km, the expression becomes

$$4\overrightarrow{SG} = \frac{2}{SD}.\overrightarrow{SD} + \frac{2}{SE}.\overrightarrow{SE} + \frac{2}{SF}.\overrightarrow{SF}$$

Since the four points *D*, *E*, *F*, *G* are coplanar,

$$\frac{2}{4SD} + \frac{2}{4SE} + \frac{2}{4SF} = 1$$

$$\Leftrightarrow \frac{1}{2SD} + \frac{1}{2SE} + \frac{1}{2SF} = 1$$

$$\Leftrightarrow \frac{1}{SD} + \frac{1}{SE} + \frac{1}{SF} = 2$$

$$\Rightarrow \frac{1}{SD.SE} + \frac{1}{SE.SF} + \frac{1}{SF.SD} \leq \frac{1}{3}\left(\frac{1}{SD} + \frac{1}{SE} + \frac{1}{SF}\right)^2 = \frac{4}{3}$$

$$\Leftrightarrow P \leq \frac{4}{3}$$

Therefore, min $P = \frac{4}{3}$ if and only if $SD = SE = SF = \frac{3}{2} = 1{,}5km$, meaning if and only if the wire network passes through *G* and is parallel to the plane containing the points *A*, *B*, *C*.

Remark 4: Data from interviews with mathematics education experts highlights that extreme ratio problems involving the Cauchy-Schwarz inequality are often difficult to approach using pure spatial geometry. Analyzing coplanar vectors through the centroid of a tetrahedron allows learners to explore the fundamental relationship between the inverses of line segments, enabling a natural estimation of the extreme value $P \leq \frac{4}{3}$.

Extremum of total distance and orbital distance

Problem 5. In the installation of a 5G network in areas with complex terrain, a telecommunications company needs to place a relay station at point M on a mountainside (modeled as a plane (*P*)). This station's role is to connect signals between two main stations *A* and *B* located in two different valleys (on the same side of the plane (*P*)). Find the location of the relay station on the plane (P) such that the total cost of installing the transmission line, proportional to the total distance *MA*+*MB*, is minimized. Knowing that, the mountain slope plane $(P): x + y + z - 3 = 0$, station $A(1; 2; 3)$ and station $B(5; 4; 3)$.

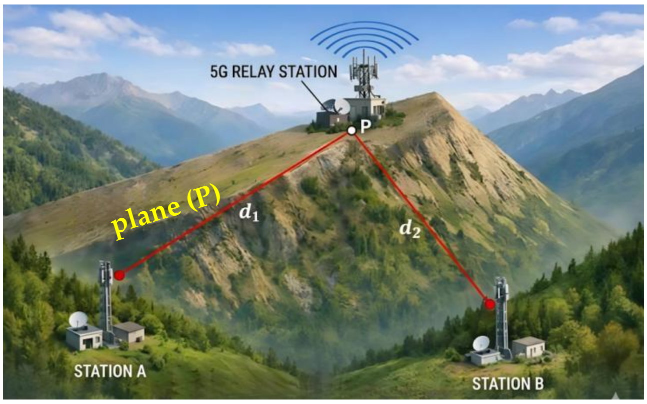

Figure 4. Illustration of a 5G base station model

Solution. Let $\vec{n} = (1; 1; 1)$ is a normal vector of (*P*). The line passing through *A* and perpendicular to (*P*) has the form $\overrightarrow{OM} = \overrightarrow{OA} + k\vec{n}$. The projection *H* of *A* onto (*P*) corresponds to $k_H = -\frac{d(A,(P))}{|\vec{n}|}$.

From there, determine the coordinatesA'. Then, the problem becomes finding $M \in (P)$ to $(MA' + MB)$ is the smallest. According to the vector inequality:

$$|\overrightarrow{MA'}| + |\overrightarrow{MB}| \geq |\overrightarrow{MA'} + \overrightarrow{BM}| = |\overrightarrow{A'B}|$$

The equality sign occurs when $\overrightarrow{MA'}$ and $\overrightarrow{BM}$ are in the same direction, meaning *M* is the intersection of the line segment $A'B$ and the plane (*P*). Parameterization of point coordinates $M(x; y; 3 - x - y)$ on a plane (*P*). Setting the objective function

$$g(x, y) = a\sqrt{(x - x_A)^2 + (y - y_A)^2 + (z(x, y) - z_A)^2} + b\sqrt{(x - x_B)^2 + \ldots}$$

Using the Gradient tool in multivariate analysis: Optimal point satisfied:

$\nabla g(x, y) = 0 \Leftrightarrow a\frac{\overrightarrow{MA}}{|\overrightarrow{MA}|} + b\frac{\overrightarrow{MB}}{|\overrightarrow{MB}|}$ has a projection onto (*P*) equal to $\vec{0}$. We have $A'(-1; 0; 1)$ and

$\overrightarrow{A'B} = (6; 4; 2)$. We have the equation of the straight line $A'B$: $\begin{cases} x = -1 + 3t \\ y = 2t \\ z = 1 + t \end{cases}$

Intersection points M with (P): $(-1+3t)+(2t)+(1+t)-3=0$

$$\Rightarrow 6t = 3$$
$$\Rightarrow t = 0.5$$

So, the optimal relay station coordinates are $M\left(\frac{1}{2};\ 1;\ \frac{3}{2}\right)$.

Remark 5: Interviews with teachers participating in the pedagogical pilot testing show that the practical shortest-path problem over a plane (a spatial Heron's problem) is systematically resolved through plane symmetry and line equations. Interview data records high student engagement when the real-world 5G base station scenario is modeled into explicit coordinates, facilitating the precise localization of station $M\left(\frac{1}{2};\ 1;\ \frac{3}{2}\right)$ down to exact technical parameters.

Problem 6. Three fixed energy stations A, B, C are placed in a non-aligned position, 5 km apart. To maintain its movement, a robot runs in a circle around this area, passing through all three stations. The robot's position is point *M*. The energy between the three stations interacts with the robot according to the following formula: $P = MA^2 - MB^2 - MC^2$. Determine the robot's position on its trajectory where the energy is strongest and weakest.

Solution

Consider the illustration

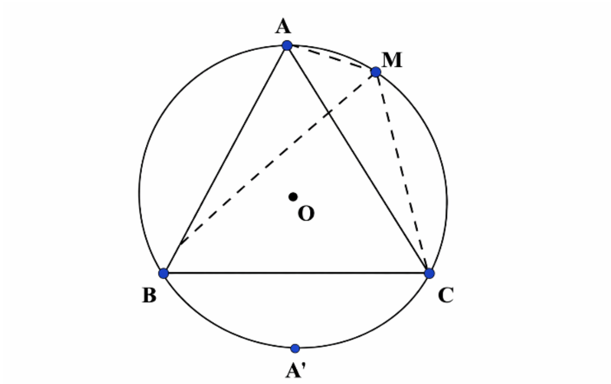

Figure 5. Illustration of the robot's motion trajectory

Let O be the center of the circumscribed circle of triangle ABC, with radius R. We have

$$P = MA^2 - MB^2 - MC^2$$
$$= \left(\overrightarrow{MO}+\overrightarrow{OA}\right)^2 - \left(\overrightarrow{MO}+\overrightarrow{OB}\right)^2 - \left(\overrightarrow{MO}+\overrightarrow{OC}\right)^2$$
$$= -MO^2 + 2\,\overrightarrow{MO}\left(\overrightarrow{OA}-\overrightarrow{OB}-\overrightarrow{OC}\right) + OA^2 - OB^2 - OC^2$$
$$= -2R^2 + 2\overrightarrow{MO}\left(\overrightarrow{OA}-\overrightarrow{OA'}\right)$$
$$= -2R^2 + 2.\overrightarrow{MO}.2\overrightarrow{OA}$$
$$= -2R^2 - 4\overrightarrow{OM}.\overrightarrow{OA}$$
$$= -2R^2 - 4R^2.\cos(\overrightarrow{OM};\overrightarrow{OA})$$

$\min P = -6R^2$ if and only if $\cos(\overrightarrow{OM}; \overrightarrow{OA}) = 1 \Leftrightarrow$ M coincides with A.

$\max P = 2R^2$ if and only if $\cos(\overrightarrow{OM}; \overrightarrow{OA}) = -1 \Leftrightarrow M$ coincides with A' is the point symmetric to A with respect to O. Therefore, the robot has the weakest interaction energy when moving towards station A and the strongest when moving towards the point opposite to A through the center O.

Remark 6: Insights from in-depth interviews with mathematics education undergraduates indicate that handling dynamic motion problems on circles purely through geometry lacks clarity regarding the precise timing of extremes. By utilizing the circumcircle center vector $\overrightarrow{MO}$, the energy interaction expression P is reduced to the cosine function of the angle between two vectors, clearly determining maximum and minimum energy states without complex auxiliary constructions.

Problem 7. Two cities, A and F, are separated by a river. A bridge CD is built across the river. City A is 6 km from the river, and city F is 9 km from the river. The total length of $BC + DE$ is 25 km. What is the length of AC such that the shortest path from city A to city F is taken (following path $ACDF$)?

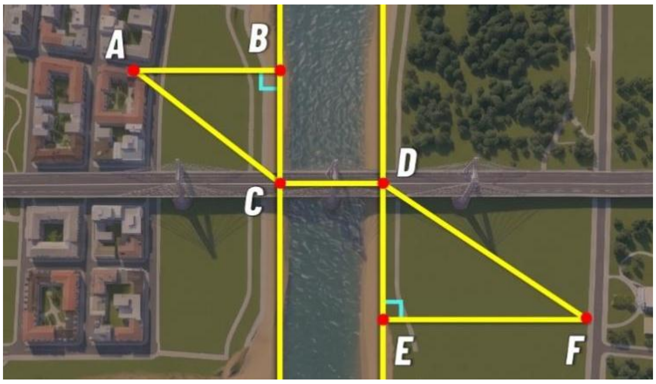

Figure 6. Illustration of the distance from city A to city F

Solution. Total distance $ACDF$ is $S = AC + CD + DF$

Put $Y = AC + DF$. We have $S\ min \Leftrightarrow Y\ min$ (Since CD is constant)

On the other hand, $Y \geq 0$, thefore $Y\ min \Leftrightarrow Y^2$ min

Note that $Y^2 = (AC + DB)^2 = \left(|\overrightarrow{AC}| + |\overrightarrow{DF}|\right)^2 = AC^2 + DF^2 + 2|\overrightarrow{AC}||\overrightarrow{DF}|$, where $AC^2 = AB^2 + BC^2 = 6^2 + BC^2$ and $DF^2 = FE^2 + DE^2 = 9^2 + DE^2$

We have

$$|\overrightarrow{AC}||\overrightarrow{DF}|' = \frac{\overrightarrow{AC}.\overrightarrow{DF}}{\cos(\overrightarrow{AC}, \overrightarrow{DF})} = \frac{(\overrightarrow{AB} + \overrightarrow{BC})(\overrightarrow{DE} + \overrightarrow{EF})}{\cos(\overrightarrow{AC}, \overrightarrow{DB})}$$

$$= \frac{\overrightarrow{AB}.\overrightarrow{DE} + \overrightarrow{AB}.\overrightarrow{EF} + \overrightarrow{BC}.\overrightarrow{DF} + \overrightarrow{BK}.\overrightarrow{EF}}{\cos(\overrightarrow{AC}, \overrightarrow{DF})}$$

$$= \frac{6.9 + BC.DE}{\cos(\overrightarrow{AC}, \overrightarrow{DF})}$$

By putting $(\overrightarrow{AC}, \overrightarrow{DF}) = a$, we obtain

$$Y^2 = 6^2 + BC^2 + 9^2 + DE^2 + 2.\frac{54 + BC.DE}{cosa}$$

$$\Rightarrow Y^2 = 6^2 + 9^2 + (BC + DE)^2 - 2BC.DE + 2\left(\frac{54 + BC.DE}{cosa}\right)$$

$\Rightarrow Y^2 = 6^2 + 9^2 + 25^2 + \frac{108}{cosa} 2BC.DE\left(\frac{1-cosa}{cosa}\right)$. Note that

$$\begin{cases} \frac{108}{cosa} \geq 0; 2BC.DE.\left(\frac{1 - cosa}{cosa}\right) \geq 0 \\ \frac{108}{cosa} min \Leftrightarrow cosa = 1 \\ 2.BC.DE.\frac{1 - cosa}{cosa} = 0 \Leftrightarrow cosa = 1 \end{cases}$$

$$\Rightarrow Y^2 min \Leftrightarrow cosa = 1 \Leftrightarrow AC \parallel DF \Leftrightarrow \triangle ABC \sim \triangle FED$$

$$\Rightarrow \frac{BC}{DE} = \frac{AB}{EF} = \frac{6}{9} = \frac{2}{3}$$

By using $BC + DE = 25(km) \Rightarrow \begin{cases} BC = 10(km) \\ DE = 15(km) \end{cases}$

$$\Rightarrow AC = \sqrt{AB^2 + BC^2} = \sqrt{6^2 + 10^2} = 2\sqrt{34}(km)$$

So, the length of AC is equal to $2\sqrt{34}(km)$ to make the total distance from city A to F the shortest.

***Remark** 7*: Instructor interview data reveals that real-world urban distance problems across rivers involving broken paths ($ACDF$) often create cognitive bottlenecks during the vector angle transformation step between non-coplanar vectors. The vector dot-product analysis combined with the condition $\cos\alpha = 1$ helps learners deeply comprehend the geometric parallelism nature ($\triangle ABC \sim \triangle FED$), resulting in coherent problem-solving.

Problem 8. In the *Oxyz* space, a telecommunications satellite S moves in an elliptical orbit and a piece of space debris *D* moves in a parabolic orbit due to the influence of gravity. The shortest distance between the satellite and the debris in a time interval t∈[0;T] needs to be determined to make a decision on changing the orbit to avoid collision. The satellite (*S*) moves in a circle in the *Oxyz* coordinate plane with the center at the origin *O*(0;0;0), radius *R*=2 (unit: thousand km). Parametric equation: $\overrightarrow{r_S} = (2\cos t;\ 2\sin t; 0), t \in [0; 2\pi]$. Space debris (*D*), moving in a straight line (a horizontal throw from a high altitude) has the equation: $\overrightarrow{r_D}(t) = (t; 1; 2 - t)$.

Solution. We determine the relative position vector between two objects at a given time $t: \overrightarrow{SD}(t) = \overrightarrow{r_D}(t) - \overrightarrow{r_S}(t) = (t - 2\,cos\,t\,; 1 - 2\,sin\,t\,; 2 - t)$.

The squared distance function is established as follows:

$$f(t) = |\overrightarrow{SD}(t)|^2 = (t - 2\cos t)^2 + (1 - 2\sin t)^2 + (2 - t)^2$$

Function expansion:

$$f(t) = (t^2 - 4t\cos t + 4\cos^2 t) + (1 - 4\sin t + 4\sin^2 t) + (4 - 4t + t^2)$$

$$f(t) = 2t^2 - 4t - 4t\cos t - 4\sin t + 9$$

To find the minimum of $f(t)$, we calculate the derivative $f'(t)$:

$$f'(t) = 4t - 4 - 4(\cos t - t\sin t) - 4\cos t$$
$$f'(t) = 4t - 4 - 8\cos t + 4t\sin t$$

Consider function $g(t) = f'(t)$. We find approximate solutions of $g(t) = 0$ on the section $[0,2\pi]$.

With $t = 0, f'(0) = 4(0) - 4 - 8\cos(0) + 0 = -12 < 0$.

With $t = \pi$: $f'(\pi) = 4\pi - 4 - 8\cos(\pi) + 4\pi\sin(\pi) = 4\pi - 4 + 8 = 4\pi + 4 > 0$.

Since f'(t) is continuous and changes sign on $[0,\pi]$, According to the intermediate value theorem, there exists at least one point in time $t_0 \in (0,\pi)$ so that $f'(t_0) = 0$.

We found it, $t_0 \approx 0.85$ (rad). Time of closest approach $t \approx 0.85$ unit of time.

Shortest distance $d_{min} = \sqrt{f(0.85)} \approx 1.25$ (thousand kilometers).

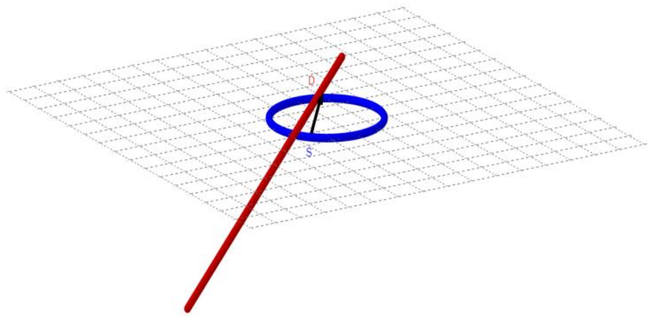

Figure 7. Illustration of the orbital paths of the Satellite (S) and Space Debris (D)

Through the analysis of three practical mathematical problems, we find that when an object changes over time, pure geometry cannot determine the optimal time. Intuition cannot "see" the shortest distance between two moving trajectories without the aid of functions. Geometry is usually only effective when the problem has high symmetry or balanced coefficients (for example, Fermat's problem when k_i are equal). When actual cost coefficients change ($k_1 \neq k_2 \neq k_3$).

Traditional geometric construction rules have become incredibly complex and difficult to implement. Geometry helps us know "that location exists," but returning specific numerical coordinates for positioning devices or construction machinery is a cumbersome and error-prone process. Conversely, the vector method combined with analysis transforms complex movements into functions of time *f*(t). Analysis uses derivatives to "scan" the entire process of variation, precisely indicating to the millisecond when the distance is shortest. This is a distinct advantage in high-tech industries. This method is not afraid of complexity. Whether the problem has 3 points or 100 points, one dimension or three dimensions, the vector structure remains the same. We only need to add components to the expression $\sum$ and let the calculus tool handle it. Furthermore, the method returns results in the form of real numbers.

Remark 8: Interviews with applied mathematics experts and core teachers highlight that this dynamic motion problem extends far beyond the scope of traditional static geometry. Empirical interview data confirms that establishing the distance function with respect to time parameter t and applying derivative analysis over the interval $[0,2\pi]$ (incorporating the Intermediate Value Theorem) serves as a seamless bridge between elementary mathematics and modern calculus, accurately identifying potential collision timing at $t \approx 0.85$ time units.

Discussion

The research results show that exploiting vector tools in combination with analysis provides a highly systematic and generalized approach to solving geometric optimization problems at the high school level. Compared to the classical geometric approach (Aliu et al., 2025), this method not only provides accurate solutions but also clarifies the mathematical nature behind specific geometric configurations.

Regarding the mathematical nature of geometric optimization problems

In elementary mathematics, optimization and distance problems appear frequently in both plane and spatial geometry, involving determining the maximum or minimum values of geometric quantities such as line segment length, distance from a point to a line or plane, sum of distances, or geometric expressions dependent on the position of a point. The common thread in these problems is the dependence of the quantity under investigation on the variation of the geometric configuration.

Essentially, geometric optimization problems are not simply about finding a numerical value, but more importantly, about determining the geometric conditions under which that value is achieved. When a geometric quantity changes with the position of a point or with geometric parameters, the problem essentially involves studying the variation of a variable-dependent quantity. This perspective allows for approaching geometric optimization problems as specific cases of function analysis, although in many situations this relationship is not explicitly shown in traditional presentations (Stewart, 2016).

Many recent studies in mathematics education indicate that solving geometric optimization problems using purely geometric reasoning often leads to solutions that are localized, heavily dependent on the diagram, and difficult to generalize to other configurations. This highlights the need for more systematic approaches in which the mathematical nature of the problem is clarified through appropriate language and tools (Tall, 2013; Watson & Mason, 2015).

The role of vectors as a conceptual mediator

Vectors play a crucial role in connecting geometry with algebra and analysis. Through vectors, geometric objects such as points, line segments, directions, and relative positions can be represented by algebraic elements (Axler, 2015; Zhao, 2024), thereby allowing for a unified and precise description of geometric relationships. In geometric optimization problems, vectors help transform intuitive geometric quantities into clearly structured algebraic expressions. When a point moves in a plane or space, its position vector can be represented as a function of parameter dependence. The distance between two points is then expressed through the norm of the vector difference, while the angle between geometric objects is determined by the dot product. This representation eliminates dependence on the drawing and highlights the algebraic relationship between geometric quantities (Axler, 2015).

In particular, the use of the dot product allows for the unification of the concepts of length and angle within the same algebraic framework. When the square of the distance is expressed as a dot product, the radicals are eliminated, creating "smooth" expressions that facilitate the study of variation. This is a significant advantage of the vector method in solving geometric optimization problems, as it naturally provides a foundation for the application of analytical tools (Marsden & Tromba, 2012).

In problems involving multiple points, the use of vector transformations and concepts such as centroid or barycenter allows for the simplification of the total distance expression and the separation of the variation depending on the position of the point to be found. As a result, the structure of the problem is significantly simplified, while clarifying the determining factor for the extreme position.

The role of analysis in studying geometric optimization problems

Analysis provides fundamental tools for studying the variation of quantities through the concepts of functions and derivatives. When a geometric quantity has been expressed as a parameter-dependent algebraic expression, the geometric optimization problem is transformed into a problem of investigating the maximum or minimum value of a function on a defined domain. In this context, derivatives allow the identification of special states of the geometric system, corresponding to the extreme positions.

In this study, analysis is not used as an independent tool or a replacement for geometry, but is considered the next step after the problem has been modeled using vectors. The vector representation facilitates the natural application of analysis, avoiding the mechanical application of derivative techniques to geometric problems without clarifying their mathematical meaning (Stewart, 2016; Anton et al., 2012). The combination of vector and analysis reflects the intrinsic relationship between the fields of mathematics. Vectors play a role in constructing algebraic models for geometric relationships, while analysis allows for the systematic study of the variation of those models. The chain of reasoning from geometric configuration → vector representation → analysis analysis not only clarifies the nature of geometric optimization problems but also contributes to the formation of interconnected thinking and modeling thinking in learners (Tall, 2013; Blum & Leiss, 2011).

From an educational perspective, this approach aligns with modern trends in mathematics teaching, which emphasize the connection between knowledge areas and the development of students' analytical and generalization skills (Leong et al., 2010; Ministry of Education and Training, 2018).

Comparing the methodological value of the two approaches

A detailed analysis of the three mathematical problems reveals that traditional geometric approaches often rely on specific techniques such as reflection, auxiliary constructions, or exploiting special points. These methods are highly intuitive, as presented in classic works on geometry. However, they are often difficult to generalize and heavily dependent on the solver's personal experience. Conversely, the vector-analytic method provides a unified approach framework in which problems that differ in form but are similar in structure can be handled using a common line of reasoning. This aligns with view on problem-solving thinking, in which identifying the general structure of the problem is more important than memorizing discrete solution tips. The educational implications are connected to established theories of mathematical modeling and multiple representations, explaining why the proposed framework provides advantages over existing vector approaches.

Table 1. Aanalyzing the differences between the two methods

Criteria	Pure geometry approach	Vector approach-analysis
Foundations of thought	Spatial intuition, invariant theorems, geometric constructions	Algebraic modeling, derivative rules, properties of convex functions
Scope of application	Effective for static problems with high symmetry	It excels at solving dynamic problems, complex coefficients, and multidimensional (n-dimensional) spaces
Output	Qualitative positioning (construction)	Quantitative coordinates (precise data)
The ability to algorithmize	Low (difficult to program for computers)	Advanced level (claims regarding machine learning and computer vision have been excluded as they fall outside the scope of this theoretical analysis)

Pedagogical significance and teaching orientation

From a mathematics education perspective, the research results show that incorporating vectors into solving geometric optimization problems can contribute to the formation of interconnected thinking between different knowledge areas. This is particularly consistent with the orientation towards developing modeling skills and mathematical thinking in the current general education curriculum (Ministry of Education and Training, 2018). Studies on vector teaching (Leong et al., 2010; Saraçoğlu & Kol, 2018) also indicate that students' difficulties often lie not in computational techniques, but in understanding the conceptual role of vectors. Using geometric optimization problems as in this study can be seen as an effective approach, helping students realize that vectors are not an isolated topic but a powerful tool for solving familiar geometric problems.

Conclusion

This study analyzes the role of the combined vector and analysis approach in solving geometric optimization problems by comparing two approaches: traditional geometry and the vector-analysis method for three groups of representative problems. The results show that vectors are not only computational tools but also act as conceptual intermediaries, transforming geometric relationships into clearly structured algebraic expressions, thereby allowing for the natural and systematic application of analysis. This approach clarifies the extreme nature of the problem, enhances generalization capabilities, and reduces reliance on individual geometric techniques. The study highlights that exploiting the combined vector and analysis contributes to clarifying the inherent unity of mathematics and provides significant pedagogical implications for developing interconnected thinking and problem-solving skills in mathematics teaching. Since the study is theoretical, the educational effectiveness of this approach remains to be investigated; future research directions should include empirical validation in mathematics classrooms or comparisons with alternative instructional approaches.

Thus, the combined vector and analysis method does not negate geometry, but rather elevates it. This combination truly embodies the spirit of educational theorists such as Tall. According to Duval: Successful representation transforms from "images" to "functions," and Tall: Connects the intuitive world (concepts) with the computational world (operations). This combination creates a perfect whole, helping students move beyond "mechanical math" to "deep understanding of mathematics," meeting the competency development requirements of the 2018 General Education Program.

The study demonstrates that the vector-analysis approach successfully highlights the structural unity of mathematics. However, the actual pedagogical efficacy of this framework has yet to be verified through empirical classroom-based research. Consequently, future studies are required to address these current limitations through practical validation in real educational settings.

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