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The epistemic path toward early algebraic knowledge: A didactical reconstruction

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Abstract

Epistemological obstacles in acquiring variables and linear equations in one variable (LEOV) remain a systemic and unresolved issue in Indonesian mathematics education. Prior research has identified their symptoms but not their origins. This study reconstructs the epistemic path towards early algebraic knowledge by integrating the Anthropological Theory of the Didactic (ATD) and Didactical Design Research (DDR). Using a qualitative hermeneutic-phenomenological approach, ATD-based praxeological analysis was applied to the responses of 120 students in Grades VII to X in West Java, Indonesia, and to their teaching modules and lesson plans, followed by the design and implementation of a didactic sequence. Four obstacles were identified: ambiguity in the meaning of variables, difficulties with algebraic operations, procedural dependency without justification, and conceptual collapse when variables appear on both sides of an equation. Each was traced to the systematic absence of the logos block in didactical transposition. After implementation, full relational justification rose from 9.4% to 65.6%. The findings affirm that this epistemic path can be traversed only through designs that deliberately construct praxeological completeness, from practical technique to logical justification, with implications for curriculum and teacher professional development.

Keywords: anthropological theory of the didactic; didactic transposition; didactical design research; early algebra; epistemological obstacles; linear equation in one variable; praxeology

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Introduction

The transition from arithmetic to algebra is among the most demanding restructurings of mathematical thought required by school mathematics: students must move from computing with fixed numbers to reasoning about relationships among unknown or varying quantities. Sfard (1991) characterized this shift as reification, the capacity to treat mathematical processes as objects that can themselves be manipulated. Without reification, variables remain enigmatic symbols, equations are executed mechanically without comprehension, and algebra becomes a procedural ritual devoid of an epistemic foundation. Early algebra, particularly the concepts of variables and linear equations in one variable (LEOV), therefore constitutes an epistemic path that must be deliberately designed; when it is not, learners become disoriented not because of cognitive limitations, but because no such path has been provided for them.

Prior research has consistently demonstrated the pervasiveness of the problem. Pittalis (2023) showed that the roots of the difficulty are established well before the formal introduction of algebra, and Blanton et al. (2018) established that such obstacles are not developmental stages that resolve naturally, but structural barriers that persist without targeted didactic intervention. Ellis and Özgür (2024) confirmed that students' difficulty in understanding literal symbols as generalizations, rather than as unknown numbers, is among the most consistent findings across cultural contexts, while Kieran and Martínez-Hernández (2022) demonstrated that, without explicit scaffolding, students remain anchored in computational strategies and do not spontaneously progress towards structural approaches grounded in equivalence.

Understanding equations as conditions of equivalence, rather than merely as procedures for determining the value of x, is a robust predictor of later success in algebra (Stephens et al., 2022). The pertinent question is therefore not only what the obstacles are, but how to design a path that transcends them, for instance by explicitly facilitating the transition from enactive to symbolic representation through the incremental construction of relational understanding at each stage (Erbilgin & Gningue, 2023).

Within the Indonesian educational system, this phenomenon is the norm rather than an anomaly. Analyses of teaching modules, lesson plans, and classroom practice across diverse contexts reveal a consistent pattern: variables are introduced as 'an unknown number', equations are solved through a transposition algorithm whose validity is never explained, and justification of solution steps has never been an integral part of classroom mathematical discourse (Jupri & Drijvers, 2016; Fardian et al., 2025; Maudy & Ruli, 2025). What students acquire is thus true belief without justification: a precarious epistemic stance that collapses when tasks deviate even slightly from those previously encountered (Pritchard, 2018). The present study examines this problem empirically through diagnostic testing of 120 students in Grades VII–X, praxeological analysis of the teaching documents used in their schools, and the design, implementation, and retrospective analysis of an alternative didactical sequence.

Two complementary frameworks were employed. The Anthropological Theory of the Didactic (ATD; Chevallard, 1985, 2006, 2019) served as the diagnostic tool: its praxeological analysis identifies precisely which components of mathematical knowledge are missing from existing didactical designs and where they are lost in the transposition process. Didactical

Design Research (DDR; Suryadi, 2019) provided the design methodology for responding to that diagnosis by deliberately constructing the complete epistemic path from practical technique to logical justification. The two are mutually necessary: ATD alone remains diagnostic, while DDR without ATD lacks the analytical precision to address root causes rather than surface manifestations.

Recent research reinforces the urgency of this agenda. Fardian et al. (2025) confirmed through a praxeological analysis of Indonesian textbooks that the absence of the technological–theoretical components in didactic transposition directly produces such obstacles, and Veith et al. (2023) showed bibliometrically that early algebra and the conceptualization of variables remain central yet unresolved themes. Radford (2022) added that designs which deny students space for independent meaning-making actively hinder the development of mathematical subjectivity. What distinguishes the present study from these investigations is the integration of ATD as a praxeological diagnostic tool with DDR as an obstacle-based design methodology, a shift from asking ‘why can students not do this?’ to asking ‘why does the system not provide a path?’, and from diagnosis to reconstruction.

This study aims to address three objectives: (1) to identify the epistemological challenges faced by students in learning variables and LEOV through ATD-based praxeological analysis; (2) to analyze the structural underpinnings of these challenges within the didactical transposition chain; and (3) to design and implement a DDR-based didactical sequence that effectively addresses the identified obstacles. In doing so, the article provides a theoretical contribution by demonstrating the integrability of ATD and DDR as a cohesive analytical-design framework. Additionally, it offers a practical contribution by presenting an epistemic didactical design that is adaptable to various early algebra learning contexts.

Methods

This study employed a qualitative approach with a hermeneutic-phenomenological orientation within the framework of Didactical Design Research (DDR; Suryadi, 2019, 2023). The phenomenological dimension was operationalized as the systematic description of students’ experiences of variables and equations as they appear in written responses and interview talk, and the hermeneutic dimension as iterative cycles of interpretation in which each student text was read, coded, contrasted with the theoretical framework, and re-read until a stable interpretation was reached (Moustakas, 1994; Fardian et al., 2024). In operational terms, every analytical claim about an obstacle was required to be traceable to (a) a verbatim student artefact, (b) an explicit coding decision, and (c) a triangulating source.

Analytical framework: DDR and ATD

DDR rests on the classical definition of knowledge as justified true belief (Pritchard, 2018): a technique that reliably produces correct answers, but whose validity a student cannot justify, does not yet constitute knowledge. Suryadi (2019) translated this into four progressively constructed levels of belief (perceptual, memorial, introspective, and a priori), which correspond directly to Brousseau’s (1997) phases of didactical situations: action, formulation, validation, and institutionalization. In this study, the four levels functioned as an observational rubric: each phase of the designed lessons, and each student justification, was coded against the level of belief it exhibited.

ATD (Chevallard, 1985, 2006, 2019; Bosch & Gascón, 2014) models every mathematical activity as a praxeology [T, τ, θ, Θ]: a task type (T), a technique (τ) for accomplishing it, a technology (θ) that justifies the technique, and a theory (Θ) that grounds the technology. T and τ constitute the praxis block ('know-how'); θ and Θ constitute the logos block ('know-why'). A praxeology deprived of its logos block reduces technique to an unjustified ritual that fails under contextual variation and cannot produce justified true belief. Didactical transposition theory, in turn, traces how knowledge moves from scholarly knowledge to knowledge to be taught, taught knowledge, and learned knowledge; the analytical concern is the epistemological reduction that occurs when logos components are eliminated along this chain because they are deemed too abstract for schooling (Bosch & Gascón, 2006; Chevallard, 2019). In these terms, the study asked one operational question: which praxeological components are present, and which are missing, at each link of the transposition chain for variables and LEOV?

As the reference point for this analysis, an Epistemological Reference Model (ERM; Bosch & Gascón, 2006) for variables and LEOV was constructed from scholarly knowledge, independently of the curriculum under examination. The ERM comprises four task types (interpretation, construction, transformation, generalization); techniques ranging from enactive to symbolic representation and standing in competition with one another; the equivalence principle of transformation as technology θ; and the equality axioms (reflexivity, symmetry, transitivity, substitution) as theory Θ. The gap between this ERM and the knowledge to be taught in the examined documents constitutes the measure of epistemological reduction in the didactical transposition.

Research procedure

The study followed the three-stage DDR cycle (Suryadi, 2019): (1) a prospective analysis of the didactical situation prior to instruction; (2) implementation of the didactical design; and (3) a retrospective analysis linking the a priori design to its actual realization. The prospective analysis drew on three data sources. First, diagnostic testing of 120 students in Grades VII–X (38, 32, 28, and 22 students, respectively) from four purposively selected schools in West Java, Indonesia (an urban public school, a peri-urban public school, a private school, and an Islamic junior secondary school), chosen to represent the main institutional contexts in which the national curriculum is enacted. Grade VII students were included because LEOV is formally introduced at that level; Grades VIII–X were included to test whether the hypothesized obstacles persist after repeated instruction, persistence being a defining property of epistemological (as opposed to ontogenic or didactical) obstacles. Second, documentary analysis of the six Grade VII mathematics teaching modules and eight lesson plans most commonly used in the participating schools. Third, semi-structured interviews with students and teachers as described below.

The diagnostic instrument consisted of eight items, two for each hypothesized obstacle category, developed from the ERM: interpreting the meaning of x in $2x + 3 = 11$; judging whether a variable may take more than one value; operating on unlike terms (e.g., simplifying $3x + 5$); solving a standard equation ($3x + 5 = 14$) and its structural modification ($14 = 3x + 5$); and solving a two-sided equation ($3x + 5 = x + 13$). Every item was accompanied by the open justification prompt: 'Explain why the step you took is valid.' The instrument was validated by two mathematics educators against the ERM and piloted with one class outside the sample. In-depth interviews (20–30 minutes, audio-recorded) were conducted with 12 students, three from each of the four response patterns identified in the test, and with four mathematics teachers of the participating classes.

Data analysis proceeded in five operational steps. First, all written responses and interview recordings were transcribed verbatim. Second, each response was coded for the presence or absence of praxeological components (T, τ, θ, Θ) following the coding protocol of Chaachoua et al. (2019, 2024); the same protocol was applied to all 214 tasks contained in the teaching modules and lesson plans. Third, each justification was classified into one of three levels: procedural (restates the steps taken), partial relational (refers to balance or sameness without generality), or full relational (explicitly invokes the equivalence principle). Both authors coded all responses independently; intercoder agreement reached Cohen's $\kappa = .86$, and all disagreements were resolved through discussion. Fourth, obstacle themes were derived hermeneutically by iteratively contrasting the coded responses with the interview transcripts until thematic saturation was reached. Fifth, the findings were triangulated across the three data sources (student responses, document analysis, and teacher interviews), and an obstacle was retained only when it appeared in all three, confirming its systemic rather than individual character.

The didactical design developed from the prospective analysis was implemented in one Grade VII class ($n = 32$) of the urban public school, whose students had participated in the diagnostic test. This class was selected because its students had completed the arithmetic prerequisites but had not yet received formal LEOV instruction, and because its schedule permitted three consecutive sessions of 2×40 minutes. The first author acted as teacher-implementer while the class teacher and the second author observed; all sessions were video-recorded and documented in field notes, and all student worksheets were collected. The design adhered to Brousseau's (1997) a-didacticity principle: during action phases the teacher refrained from technical intervention, so that feedback came from the milieu itself. The retrospective analysis was conducted in three concrete steps (Suryadi, 2019): (1) the anticipated student responses formulated in the a priori analysis were compared, phase by phase, with the responses that actually occurred, using the video records and worksheets; (2) students' justifications on the end-of-sequence epistemic assessment were coded with the same three-level rubric and compared with their diagnostic-test results; and (3) discrepancies between design and realization were classified as either acceptable adaptations or revision points against the three DDR design principles (coherence, unity, flexibility).

Results

Praxeological analysis: Diagnosing the blockage of the epistemic path

The ATD-based praxeological analysis of the diagnostic data revealed four interdependent epistemological obstacles stemming from a single structural root: praxeological incompleteness in the available didactical designs. Table 1 summarizes the distribution of the response categories indicating each obstacle across the 120 students. Two features of this distribution deserve emphasis. First, the obstacles were pervasive: none of the 120 students accepted that a variable may represent more than one value, 70.0% violated equivalence when operating on equations, and only 8.3% could justify their own solution steps relationally. Second, the obstacles were persistent: failure on the two-sided equation declined only modestly from Grade VII (89.5%) to Grade X (77.3%) despite three additional years of instruction. This is precisely the persistence that characterizes epistemological, rather than ontogenic or didactical, obstacles.

Table 1. Distribution of student responses indicating each epistemological obstacle (N=120)

Obstacle	Response category (diagnostic item)	n	%
O1: Meaning of variable	‘x is just a letter used in algebra’ / no meaning	52	43.3
	‘x is one unknown number to be found’	68	56.7
	Accepts that x may represent more than one value	0	0.0
O2: Algebraic operations	Concatenation of unlike terms (e.g., 3x + 5 = 8x)	74	61.7
	Operation applied to one side of an equation only	84	70.0
O3: Procedural dependency	Correct answer on standard item (3x + 5 = 14)	98	81.7
	Correct answer on modified item (14 = 3x + 5)	42	35.0
	Full relational justification of own steps	10	8.3
O4: Two-sided equation (3x + 5 = x + 13)	Refused to attempt the item	33	27.5
	Unsystematic trial-and-error substitution	37	30.8
	Isolation algorithm violating equivalence	30	25.0
	Correct solution	20	16.7

Note. Percentages are computed over all 120 participants. Categories within an obstacle are not mutually exclusive across items, because two diagnostic items probed each obstacle.

Obstacle 1: Vacuity of Variable Meaning. When students were asked to explain the meaning of ‘x’ in the equation 2x + 3 = 11, only two response categories emerged. More than two-fifths of the students (52 of 120; 43.3%) perceived the symbol ‘x’ as devoid of any specific conceptual meaning, serving solely as a conventional element of algebraic notation, with written responses such as ‘algebra problems always have x in them’ and ‘x is just the letter used in algebra problems.’ The remaining 68 students (56.7%) perceived x as a single unknown number. While not incorrect in specific contexts, this conception severely restricts the understanding of variables: when subsequently asked whether x could represent more than one number in a particular mathematical context, not a single student (0 of 120) responded affirmatively, neither in the context of equations with multiple solutions nor in the context of generalization.

In-depth interviews consistently traced the origins of this obstacle to prior learning experiences. The following excerpt (S-09, Grade VIII) is representative:

Researcher: 'In 2x + 3 = 11, what does the x stand for?'
S-09 : 'It is just the letter for algebra. Every algebra problem has an x in it.'
Researcher: 'Could x stand for other numbers as well?'
S-09 : 'No. x is only the answer. Here the answer is 4, so x is 4.'

Other students referred directly to instruction: 'My teacher said x is the number we are looking for' (S-03, Grade VII), and 'the first page of the module says a variable is a substitute for a number that is not yet known' (S-11, Grade IX). Analysis of the six teaching modules confirmed this account: not one introduced variables in a context of generalization or covariational relationships; all introduced variables exclusively as the 'unknown' in an equation with a single value to be determined. From an ATD perspective, the task types T available in the existing design did not encompass those required for a fuller understanding of variables: a single task type dominated (finding the value of x), with no variation fostering variables as general representations.

The epistemological implications are profound. If a variable lacks any significance beyond its numerical value, an algebraic expression such as 3x + 5 cannot be comprehended as an independently meaningful mathematical entity. Its meaning is confined to the context of a specific equation with a solution. This directly impedes the development of structural algebraic reasoning, which is the primary objective of school algebra (Kieran, 2004; Radford, 2021).

Obstacle 2: Difficulty with Algebraic Operations. Two primary error patterns were identified. First, the concatenation of unlike terms, produced by 74 students (61.7%): responses such as 3x + 5 = 8x treat 3x and 5 as homogeneous terms that can be combined by ordinary arithmetic, indicating that the concept of x as an abstract symbol of a different dimension from a constant had not been developed. One representative written response read: '3x + 5 means 3x and 5, so together it is 8x, because 3 + 5 = 8' (S-22, Grade VII). Second, and epistemologically more significant, the failure to preserve equivalence, exhibited by 84 students (70.0%): in equations involving variables on both sides, these students applied an operation to only one side, or applied different operations to each side, without recognizing the violation of the equivalence principle.

Praxeological analysis of the teaching documents confirmed the structural root of this problem. Across the 214 tasks coded in the six modules and eight lesson plans, not one contained a technological discourse θ explaining the mathematical validity of an equation transformation. The technique of 'transposing a term with a change of sign' was presented as a rule without justification: 'If a positive term on the left side is to be moved to the right, its sign becomes negative.' No reference was made to the principle that subtracting the same value from both sides preserves equivalence, nor to the equality axioms as theory Θ. The logos block of the praxeology of equation transformation was entirely absent.

Obstacle 3: Procedural Dependency Without Epistemic Justification. The contrast between procedural competence and epistemic understanding was stark. Of the 120 students, 98 (81.7%) answered the standard item $3x + 5 = 14$ correctly, but only 42 (35.0%) could solve the structurally modified item $14 = 3x + 5$, although the two are mathematically identical. S-17 (Grade VII) is representative: having solved the standard item flawlessly, the student wrote for the modified item, 'This one cannot be solved, because the x is on the wrong side.' When asked to justify their initial step on the standard item ('Why is this step valid?'), 87 students (72.5%) produced purely procedural restatements ('because the 5 moves to the right and becomes minus'), 23 (19.2%) produced partial relational justifications, and only 10 (8.3%) invoked the equivalence principle in full relational form.

From the DDR perspective, this exemplifies a paradigmatic instance of learning that achieves only true belief without justification. Students hold the belief that a specific procedure yields correct answers (a true belief grounded in repeated experience), but they lack the epistemic justification explaining the procedure's validity. Consequently, students' adaptive capacity, or their ability to adjust techniques when the surface structure of a task changes slightly, is severely compromised. This represents a specifically incomplete praxeology: τ is accessible, but θ and Θ are not, resulting in students' limited epistemic access to the scope and conditions of use of the techniques they possess.

Obstacle 4: Conceptual Collapse with Two-Sided Equations. When faced with the equation $3x + 5 = x + 13$, only 20 students (16.7%) reached a correct solution. The remainder exhibited three distinct responses: 33 students (27.5%) refused to attempt the problem, 37 (30.8%) conducted unsystematic trial-and-error substitution, and 30 (25.0%) applied an isolation algorithm that disrupted the equivalence structure. Interviews revealed a highly consistent implicit model underlying these behaviors: students conceptualized equations as objects with a 'problem side' (where the expression resides) and a 'result side' (where the answer is expected to appear). The following exchange (S-14, Grade VII) makes the schema explicit:

Researcher: 'Why did you stop working on $3x + 5 = x + 13$?'

S-14 : 'Because there is an x on the right side too. The right side should be where the answer comes out. There is no place to put the answer.'

This schema, functional and indeed correct in the context of standard equations, became an epistemological impediment when the equation's structure lacked an 'empty side for the answer.'

This constitutes an epistemological obstacle in the strictest Brousseauian sense: it is not an error resulting from insufficient learning, but an error caused by having acquired an overly specialized schema that is valid within a restricted context but hinders generalization. The equation schema with a 'result side' represents an overly hasty generalization from limited experience with standard equations, functioning effectively until it encounters a different structural pattern. This characteristic distinguishes epistemological obstacles from errors caused by insufficient knowledge: they arise from existing knowledge rather than its absence.

The praxeological analysis of the teaching modules and lesson plans, in comparison to the ERM constructed for the domain of variables and LEOV, yielded a consistent account of three patterns of epistemological reduction in the operational didactical transposition.

First, the reduction of task types: of the four task types encompassed in the ERM (interpretation, construction, transformation, generalization), the available designs almost exclusively addressed a single type. Of the 214 tasks coded, 187 (87.4%) were algorithmic transformations (finding the value of x in an already-formed equation through the isolation technique), while construction tasks (building an equation from a situation) accounted for 14 (6.5%), interpretation tasks (making sense of an algebraic expression) for 13 (6.1%), and generalization tasks (formulating patterns using variables) for none (0.0%). The direct consequence of this reduction is a severely narrow praxeology: students possess only a single task type in their algebraic schema, such that any variation from it immediately becomes a source of difficulty.

Second, the reduction of techniques: a single dominant technique without rivals. No alternative techniques were introduced; no contexts were provided in which a different technique would be more efficient or more meaningful. From the perspective of praxeological dynamics (Chaachoua et al., 2019, 2024), the absence of technique competition is the condition that prevents the development of technological discourse: where only one technique exists, there is no motivation to ask ‘why does this technique work?’ because no alternative technique questions or challenges it.

Third, and most critically: the near-total absence of the logos block. Of the 214 tasks, only nine (4.2%) contained even a partial reference to technology θ (the equivalence principle justifying transformation), and none articulated theory Θ (the equality axioms). Teacher interviews confirmed that this absence was deliberate, arising not from oversight but from institutional conviction: ‘the theory is too abstract for children of this age’ (T-02), and ‘what matters is that they can work through the examination questions’ (T-04). This constitutes the self-preserving mechanism of institutional conditions and constraints that prevents the emergence of a more complete praxeology: the dominant procedural model is maintained by the examination system, teachers’ professional beliefs, and the culture of teaching modules, collectively constituting an epistemic ecology hostile to logos. Figure 1 presents the ERM diagram identifying the obstacles and their praxeological roots as revealed through the ATD analysis.

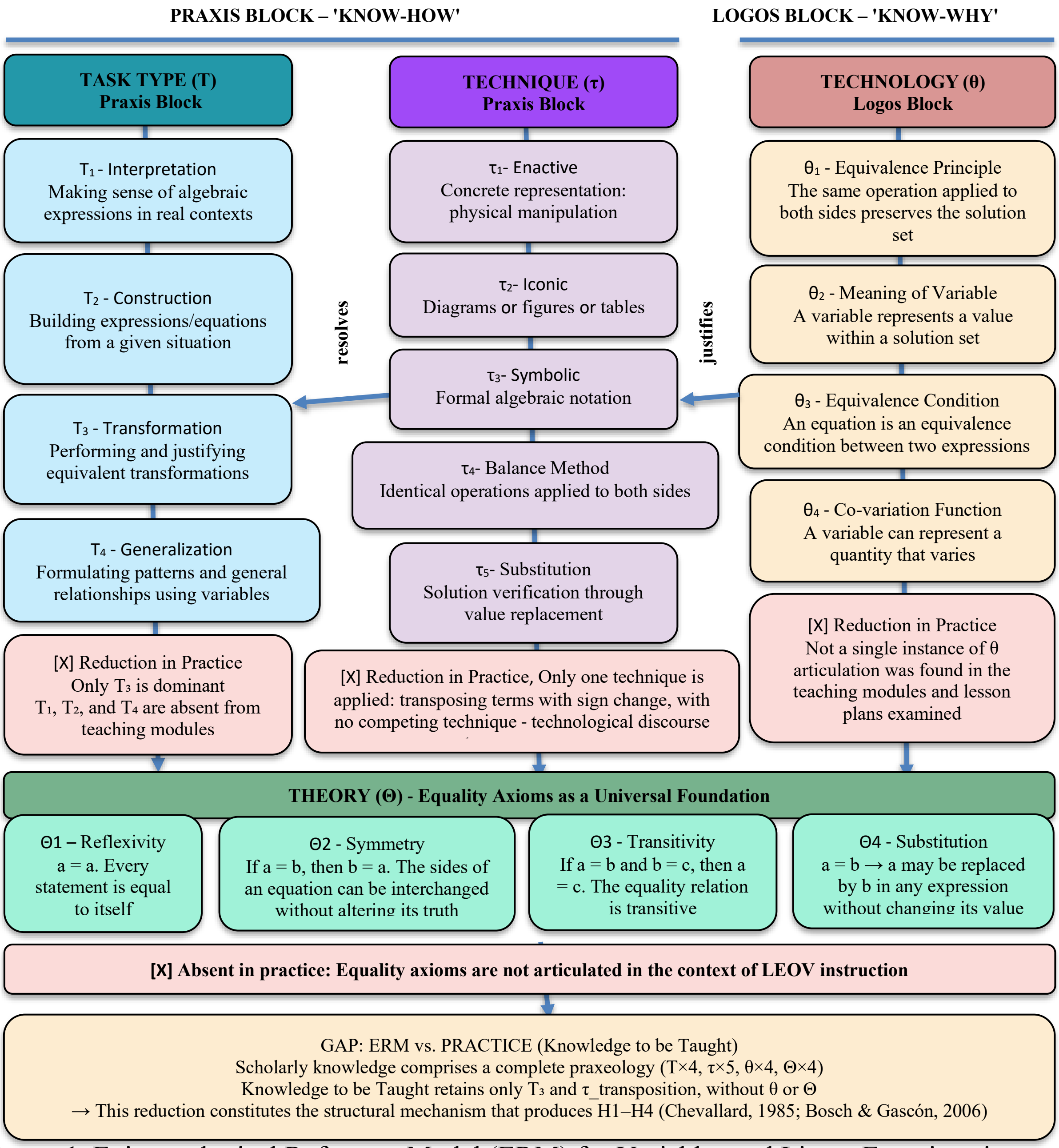

Figure 1. Epistemological Reference Model (ERM) for Variables and Linear Equations in One Variable (LEOV)

Didactical design: Constructing the epistemic path

Based on the findings of the praxeological analysis, a didactical design was developed that comprises three coherently structured didactical situation contexts. Each context constructs a distinct layer of the praxeology, forming a single epistemic arc from perceptual experience to justified true belief. Table 2 presents the epistemic roadmap of the three contexts.

Table 2. Epistemic roadmap of the three-context didactical design

Dimension	Context 1 Meaning of Variable	Context 2 Equation as Equivalence	Context 3 Equivalence as Theory
Key Obstacles	H1: Vacuity of variable meaning	H2 & H3: Operations without justification	H3 & H4: Procedural dependency; conceptual collapse in two-sided equations
Praxeological Components	Diverse T + τ (enactive→symbolic)	τ + emerging θ (physical balance)	Explicit θ + Θ (equality axioms)
DDR Belief Level	Perceptual + Memorial	Memorial + Introspective	Introspective + A Priori
TDS Phase (Brousseau, 1997)	Action & Formulation	Formulation & Validation	Validation & Institutionalization
Primary Materials	Physical measuring cups + Exploration worksheet	Cup arrangement + Comparison worksheet	Formalisation worksheet + Principle cards
Epistemic Output	Variable as a meaningful representational tool	Equation as an equivalence condition between two expressions	Principle and axioms as justified true belief

In Context 1, each group of students worked with a set of six measuring cups of three types (three cups of type A, two of type B, and one of type C) whose volumes were initially unknown to the students. The situation was chosen because it makes the inadequacy of purely arithmetic representation physically experienceable: as long as the volumes are unknown, the total volume can only be expressed by naming the cup types, and this is precisely the epistemic need from which the variable emerges as a solution rather than as an imposed convention. Table 3 presents the flow of didactical situations and the praxeological analysis for Context 1.

Table 3. Flow of didactical situations and praxeological analysis, context 1: building the meaning of variable

Phase	Didactical Activity	Epistemic Function & Praxeological Components
Action Situation (±15')	Task: "Determine the total volume of all cups. Record your method in as much detail as possible." Anticipated responses: (a) direct enumeration (40+40+…); (b) additive grouping; (c) multiplicative grouping (3×40 + 2×30 + 1×10). Teacher observes without technical intervention; identifies groups that produce multiplicative responses.	Builds perceptual belief (DDR). A-didacticity maintained: the milieu provides direct feedback independent of the teacher (Brousseau, 1997). Task types T_1 (interpretation) and T_2 (construction) begin to form through enactive experience (τ_1).
Formulation Situation (±20')	Task: "Represent your solution in at least three different ways." Critical moment: "What if we do not know how much each cup holds? Could we still write the total volume?"	Builds memorial belief (DDR). Variable emerges as a solution to the inadequacy of arithmetic representation, providing the condition that enables reification (Sfard, 1991).

Phase	Didactical Activity	Epistemic Function & Praxeological Components
	Teacher elicits the notation 3a + 2b + c organically.	Techniques τ_2 (iconic) and τ_3 (symbolic) form; T_3 generalization begins to activate.
Validation Situation (±25')	Teacher reveals: A = 40 ml, B = 30 ml, C = 10 ml. "Substitute into 3a + 2b + c. Is the result consistent?" Extension: "If a were changed to 50 ml, what would the total be?" and "Can a represent more than one value?" Informal institutionalization: variable as a representation for any given value.	Deeper memorial belief: students verify that the symbolic expression is equivalent to enumeration. Task type T_4 evaluation forms (τ_4 substitution). Extension questions open the co-variational function of variable, transcending the unknown function underlying H1. Partial θ begins to form.
Praxeology Constructed	T: T_1 Interpretation, T_2 Construction, T_3 Generalization, T_4 Evaluation τ: τ_1 Enactive (summation per cup), τ_2 Iconic (multiplicative grouping), τ_3 Symbolic (3a+2b+c), τ_4 Substitution	Partial θ: the variable represents an undetermined or variable value, not a hidden number. Θ: Not yet articulated (developed in Context 3). Foundation laid: validity of the symbolic expression can be empirically verified.

In the context of Context 2, students were presented with a scenario where two distinct cup arrangements were recognized to have an equal total volume. Specifically, two Type A cups and one Type B cup (left tray) were equivalent to one Type A cup and three Type B cups (right tray). The utilization of two trays labeled 'Left Side' and 'Right Side' facilitated the tangible representation and manipulation of the concept of 'both sides' before its abstract formulation. The pivotal moment in the development of technology θ transpired when the instructor simultaneously removed one Type A cup card from both trays: "I am removing one Cup A from both sides simultaneously. Are both trays still equivalent? Why?" This physically verifiable action established the groundwork for the equivalence principle: "Performing the same action on both sides maintains the total equivalence." Table 4 outlines the sequence of didactic scenarios presented in Context 2.

Table 4. Flow of didactical situations and praxeological analysis, context 2: Equation as an equivalence condition

Phase	Didactical Activity	Epistemic Function & Praxeological Components
Action Situation (±15')	Setup: Left tray = 2A + 1B; Right tray = 1A + 3B; both trays are equal in total volume. Task: "Write an equation representing this situation. Determine the volumes of A and B and show how you know your answer is correct." Anticipated responses: 2a + b = a + 3b (symbolic); trial substitution; or physical-intuitive strategy.	The transition from physical situation to formal representation reinforces memorial belief: the variable from Context 1 is now actively used to construct an equation. Task type T_1 (representation of an equation from a concrete situation) forms. Competing strategies begin to emerge (Chaachoua et al., 2019).

Phase	Didactical Activity	Epistemic Function & Praxeological Components
Formulation Situation (±25')	Critical moment: teacher simultaneously removes one cup A card from Both trays: "Are both trays still equivalent? Why?" Left → 1A + 1B; Right → 3B → A + B = 3B → A = 2B. Teacher facilitates student formulation of the principle: "Performing the same operation on both sides preserves the equivalence of the total."	Critical moment in building θ: the equivalence principle is constructed from physically verifiable experience (phenomena) before it becomes a mathematical principle (noumena). Competition between τ_1 (physical balance) and τ_2 (elimination) drives the emergence of technological discourse (Chaachoua et al., 2019, 2024).
Validation Situation (±20')	New task: "Solve 3a + 2b = a + 4b + 10. Write each step together with its justification." Scaffolding: "What can you do to both sides? Is equivalence still maintained?" Written reflection: "Write the principle you used in one sentence."	Validation situation of TDS (Brousseau, 1997) → introspective belief (DDR): students prove their answer to themselves through physical checking. Writing the principle initiates the institutionalization of θ, produced collectively rather than transmitted by the teacher. H2 and H3 begin to be addressed at the epistemic level.
Praxeology Constructed	T: T_1–T_4 (Context 1) + T_5 Solving via balance method τ: τ_1 Physical balance + τ_2 Elimination (active competition)	θ becoming explicit: "The same operation applied to both sides preserves the equivalence condition of the equation." Θ introduced informally: equality axioms as the foundation of θ begin to be articulated.

In the third context, equations with variables on both sides were introduced, building upon the cup situations established in the first and second contexts. During the reflection and abstraction phase, students were tasked with formulating a single sentence encapsulating the principle they consistently applied; the teacher then facilitated the convergence of these student formulations towards a formal institutionalization of θ: "The same operation applied to both sides preserves the solution set of the equation." This institutionalization was initiated from students' own experiences rather than from teacher explanations, distinguishing it from the persuasive-testimonial character of knowledge transmission (Suryadi, 2019). Subsequently, the institutionalization of theory Θ introduced the equality axioms (reflexivity, symmetry, transitivity, substitution) as the mathematical foundation ensuring the universal validity of the equivalence principle, thereby completing the epistemic circle from phenomena (cup experience) through noumena (inference of the equivalence principle) to formal knowledge (equality axioms as theory Θ). Table 5 presents the complete didactical situation flow and praxeology for Context 3.

Table 5. Flow of didactical situations and praxeological analysis, context 3: Institutionalization of the equivalence principle as theory

Phase	Didactical Activity	Epistemic Function & Praxeological Components
Reflection & Abstraction (±20')	Task: "From Worksheet 2, what principle did you always apply? Formulate it in one sentence." Teacher displays students' various formulations without evaluating them, then facilitates convergence. Institutionalization of θ: "The same operation applied to both sides preserves the solution set of the equation."	Institutionalization of θ is initiated from students' own experience rather than through teacher transmission, distinguishing it from the persuasive-testimonial character (Suryadi, 2019). Introspective belief (DDR): students articulate θ from their experience in Contexts 1 and 2.
Validation: Two-Sided Equations (±25')	Task: "Solve $3x + 5 = x + 13$. Show every step together with its justification." Contextual anchor: "Which cup type appears on both sides?" Mandatory format: Step: […] | Justification: […]. The Equivalence Principle applies. Variation tasks: $5x - 3 = 2x + 9$; $4(x + 1) = 2x + 10$; $7 = 3x - 2$.	H4 addressed: the source-side/result-side schema is dismantled through two-sided equations. Mandatory justification format actively renegotiates the didactical contract. Active technique competition: students select the most efficient technique, driving the development of technological discourse (Chaachoua et al., 2024).
Institutionalization of Θ & Epistemic Assessment (±15')	Institutionalization of Θ using the Equality Axioms: Reflexivity ($a = a$); Symmetry ($a = b \rightarrow b = a$); Transitivity ($a = b, b = c \rightarrow a = c$); Substitution ($a = b \rightarrow a$ may be replaced by b). Independent assessment: "Solve $4x - 7 = x + 8$. Write every step and its justification." Assessment criteria: (1) correct answer; (2) every step references θ; (3) language demonstrates relational understanding.	Institutionalization of Θ closes the epistemic circle: τ is justified by θ; θ is justified by Θ. A priori belief (DDR): the praxeology of linear equations is now complete.
Complete Praxeology after Context 3	T: T_1 Interpretation, T_2 Construction, T_3 Transformation + justification, T_4 Generalization, T_5 Verification, T_6 Two-sided equations τ: τ_1 Enactive–physical, τ_2 Elimination, τ_3 Isolation, τ_4 Substitution (active competition; students choose based on efficiency)	θ: Equivalence Principle: "The same operation applied to both sides preserves the solution set of the equation." Used as explicit justification at every transformation step. Θ: Equality Axioms: Reflexivity, Symmetry, Transitivity, Substitution. Θ justifies θ universally and connects praxis to logos within a coherent framework.

Metapedadidactical analysis and retrospective findings

Metapedagogical analysis conducted during the implementation of the three contexts confirmed the adherence to all three design principles of the DDR framework. The coherence principle was met, as Context 2 consistently drew upon the representations and language constructed in Context 1, and Context 3 consistently drew upon the principles constructed in Context 2. The unity principle was also satisfied, as the entire three-context sequence was oriented towards a single epistemic objective: constructing the comprehensive praxeology of LEOV, encompassing diverse task types, explicit technology, and theory. The flexibility principle was partially fulfilled, as the researchers required adaptations to accommodate unanticipated student responses, particularly the emergence of hybrid strategies that combined enactive techniques with symbolic notation in ways not explicitly designed.

The retrospective analysis then compared students' epistemic condition before implementation, as documented in the diagnostic test, with their condition after the three contexts, as documented in the end-of-sequence epistemic assessment ($4x - 7 = x + 8$ with mandatory justification). Table 6 summarizes the shifts on seven indicators for the implementation class ($n = 32$).

Table 6. Students' epistemic condition before and after implementation of the didactical design ($n = 32$)

Indicator	Before n (%)	After n (%)
Correct solution of a two-sided equation	6 (18.8)	26 (81.3)
Full relational justification	3 (9.4)	21 (65.6)
Partial relational justification	6 (18.8)	6 (18.8)
Procedural-only justification	23 (71.9)	5 (15.6)
Explicit reference to the equivalence principle	2 (6.3)	23 (71.9)
Accepts that a variable may represent more than one value	0 (0.0)	28 (87.5)
Concatenation of unlike terms	19 (59.4)	4 (12.5)

Note. 'Before' = diagnostic test administered during the prospective analysis; 'After' = end-of-sequence epistemic assessment. Justification levels were coded with the same three-level rubric on both occasions.

Three changes are noteworthy. First, the dominant reasoning type shifted from procedural to relational: before implementation, 23 of the 32 students (71.9%) could only restate their steps ('the 7 moves to the right and becomes plus'), whereas afterwards 21 students (65.6%) justified their steps relationally, in linguistically varied but semantically consistent formulations such as 'because if we subtract the same quantity from both groups, the total remains unchanged' and 'because the equation must stay balanced on both sides.' Second, the epistemological obstacles themselves receded: concatenation errors fell from 19 to 4 students, 28 students (87.5%) came to accept that a variable may represent more than one value (compared with none before), and the 'result-side' schema no longer prevented 26 students (81.3%) from solving a two-sided equation. Third, the change was not universal: five students (15.6%) remained at a purely procedural level, indicating that a three-session sequence reopens, but does not by itself complete, the epistemic path.

The evolution of individual students illustrates these shifts concretely. On the diagnostic test, S-17 had written for $14 = 3x + 5$: 'This one cannot be solved, because the x is on the wrong side.' On the epistemic assessment, the same student solved $4x - 7 = x + 8$ in three documented steps ('subtract x from both sides', 'add 7 to both sides', 'divide both sides by 3'), each accompanied by the justification 'the same operation on both sides keeps the equation equivalent', and concluded with $x = 5$. What changed is not merely the correctness of the answer but the epistemic status of the steps: from moves whose legitimacy resided in the teacher's authority to transformations justified by an explicit principle.

Discussion

The findings converge on a single assertion: the epistemological obstacles that students face in learning variables and LEOV are not individual cognitive shortcomings, but systemic outcomes of an epistemic path blocked by non-epistemic didactical design. The empirical basis for this assertion is the alignment of three independent data sources: the near-universal restriction of variable meaning and the 83.3% failure rate on the two-sided equation among the 120 students; the virtual absence of the logos block from all 214 tasks in the teaching documents; and the teachers' explicit statements that this absence is intentional. The assertion is also constructive: obstacles rooted in design can be mitigated through improved design, as the before–after shifts in Table 6 demonstrate. This section discusses the implications from four perspectives and closes with the limitations of the study.

Obstacles as systemic products, not cognitive failures

Suryadi (2019) proposed a precise formulation: a didactic design that lacks epistemological justification prior to implementation poses students at epistemic risk. When a teacher instructs the technique of transposing a term without the technology (θ) that justifies it, the teacher not only omits a useful explanation but actively disrupts the epistemic path students must traverse to attain justified true belief. This is fundamentally distinct from students who have not studied diligently enough: these are students confronted with a design that is structurally incapable of producing knowledge.

This perspective also has an ethical dimension (Radford, 2021, 2022): students who possess only true belief without justification, as 72.5% of the diagnostic sample did, are not equipped to participate fully in meaningful mathematical discourse. Recent international evidence reinforces this reading. Pitta-Pantazi et al. (2025) showed that arithmetic structure sense, though predictive of later algebraic success, is by itself insufficient without the analytical reasoning that explicit justification supplies, while Sumpter et al. (2026) documented that Grade 6 students in Sweden commit the same mathematical-equivalence errors identified here. That obstacles of identical structure recur across markedly different curricular systems (Ellis & Özgür, 2024; Kieran, 2022; Pitta-Pantazi et al., 2025; Sumpter et al., 2026) indicates that the 70.0% equivalence violations and the 83.3% failure on the two-sided equation observed in our sample reflect a shared didactical-structural deficit rather than a local or individual one.

Integrating ATD and DDR: From diagnosis to design

The integration of ATD and DDR, as proposed in this study, offers a theoretically coherent solution to the problem. ATD serves as a practical diagnostic tool, precisely identifying the missing components of the praxeology in the existing design and pinpointing the specific point in the transposition chain where these components are lost. DDR, on the other hand, provides an epistemic design methodology, enabling the construction of an alternative design justified not solely by pedagogical considerations but also by a comprehensive analysis of learning obstacles and scholarly knowledge. This complementarity also answers a recent critique. Abou-Hayt (2026) argued that ATD, taken on its own, remains largely descriptive and offers limited guidance for constructing instruction, and Strømskag and Chevallard (2026) demonstrated that moving from diagnosis to a reconstructed knowledge to be taught requires an explicit archeorganisation of the target praxeology. In the present study, DDR supplies precisely this constructive step, converting the ATD diagnosis of the absent logos block into the three-context design that reinstated technology θ and theory Θ.

Vygotsky (1978) provided the psycholinguistic foundation for this process. Scientific concepts, including formal justification, cannot develop in isolation from concrete experience; they must be constructed through a zone of proximal development bridged by appropriate instruction. Within this framework, the three design contexts developed in this study represent three structured zones of proximal development: Context 1 bridges arithmetic experience and algebraic representation (ZPD 1); Context 2 bridges operational technique and understanding of the equivalence principle (ZPD 2); and Context 3 bridges intuited principle and articulated theory (ZPD 3). Bruner's (1977) progression from enactive through iconic to symbolic representation proved to be an effective epistemic bridge in all three contexts.

The study by Winsløw et al. (2014) on study and research courses as an epistemological model in didactics provides theoretical support for this approach. They demonstrated that meaningful mathematics learning necessitates structures that not only enable students to study techniques but also allow them to explore the questions that motivate the emergence of those techniques. The three contexts in our design represent an adapted version of this principle: each context commences with a question or situation that creates an epistemic need, rather than with a technique to be learned. The efficacy of such deliberately structural designs is not unique to this setting: Kullberg et al. (2024) reported comparable gains when a structural approach replaced a procedural one in early arithmetic, an international parallel to the rise in full relational justification from 9.4% to 65.6% observed here.

Praxeological dynamics and technological discourse

The findings on praxeological dynamics (Chaachoua et al., 2019, 2024) provide a crucial additional dimension. The development of mathematical understanding is closely associated with the presence of technique competition: when students encounter multiple competing techniques for the same task type, they are motivated to develop a technological discourse that compares and explains them. The documentary data of this study show the obverse: in a design offering a single dominant technique, as 87.4% of all coded tasks did, there is no competition, no need for technological discourse, and the logos block remains unoccupied.

The progressive movement in our design contexts, from enactive techniques such as physical balance to formal techniques such as the isolation algorithm, creates conditions conducive to productive technique competition. In such environments, students can compare the efficiency and explanatory completeness of the various techniques they employ. Kaspary et al. (2020) further elaborated on this concept by introducing the notion of the 'scope of a technique' (portée de la technique): techniques with a broader scope, applicable to a wider range of task types, tend to be supported by more explicit and articulable technology. While the physical balance technique, despite its limited formal scope, possesses an epistemic advantage in that it can be readily justified through direct experience, making it an effective stepping stone towards more formal techniques.

Didactical contract and epistemic ecology

The didactical contract cannot be disregarded. Interview data consistently revealed that students' procedural orientation is not merely a cognitive preference but a learned response to a system of institutional expectations. Students comprehend that mathematics demands the presentation of solution steps rather than the explanation of their validity. This non-epistemic didactical contract operates as a second layer of impediment on the epistemic path: even if the design structurally provides material for the construction of justification, the prevailing contract prevents students from drawing upon that material.

Therefore, design reform must be accompanied by a renegotiation of the didactical contract, specifically a modification of what is recognized as 'legitimate mathematical activity' within the classroom. In our design, this renegotiation was explicitly implemented through the worksheet's 'Step | Justification' column format, which mandates justification as an integral component of the solution rather than an optional addition. Dubinsky (2001) conceptualized an analogous process through the APOS framework (Action–Process–Object–Schema): when justification becomes an integral component of the definition of a 'complete solution,' students undergo a gradual internalization of the new epistemic norm.

Chevallard (2019) and Bosch and Gascón (2006) employed the concept of the ecology of knowledge to describe the conditions that facilitate or impede the emergence of specific forms of knowledge within an institution. In an epistemic ecology that is hostile to logos, where the examination system, teachers' professional beliefs, and the culture of teaching modules collectively exert pressure towards the dominant procedural model, an epistemic didactical design can function as an 'ecological reserve': a protected space in which logos knowledge has an opportunity to flourish despite institutional pressure. The shifts documented in Table 6 indicate that such a reserve is practically feasible, not merely theoretical.

Limitations of the study

Several limitations bound the claims of this study. First, the didactical design was implemented in a single class of 32 students over three sessions; the reported shifts therefore demonstrate feasibility rather than generalizable effect, and no delayed assessment was administered to establish the stability of the epistemic reorganization over time. Second, the first author acted as teacher-implementer, which secured fidelity to the design but introduces the possibility of implementer bias; implementation by teachers not involved in the design remains to be studied. Third, the diagnostic sample, although spread across four school types and four grade levels, was drawn from a single province and is not statistically representative; the percentages reported here characterize the sample, not the national population. Fourth, the analysis is qualitative and interpretive: the before–after comparison is not a controlled experiment, and rival explanations for the observed shifts, such as novelty effects and increased attention, cannot be excluded by design. These limitations define the scope within which the conclusions should be read and motivate the research agenda outlined in the Conclusion.

Conclusion

This study set out to identify the epistemological obstacles that students face in learning variables and LEOV, to trace their structural roots in the didactical transposition, and to design a didactical response. The empirical findings can be summarized concretely. Among 120 students in Grades VII–X, none accepted that a variable may represent more than one value; 70.0% violated equivalence when operating on equations; only 8.3% could justify their own solution steps relationally; and 83.3% failed on an equation with variables on both sides. Praxeological analysis of the 214 tasks in the teaching documents located the root of these four obstacles: 87.4% of the tasks belonged to a single type, and the logos block (technology θ and theory Θ) was almost entirely absent (4.2% and 0.0% of tasks, respectively). The obstacles are thus systemic products of an incomplete praxeology transmitted through the didactical transposition, not cognitive limitations of individual students.

The DDR-based didactical design developed in response demonstrated that this blocked path can be reopened. After three sessions structured around diverse task types, competing techniques, technology θ constructed from concrete experience, and theory Θ articulated through meaningful institutionalization, full relational justification rose from 9.4% to 65.6% of the implementation class, success on two-sided equations rose from 18.8% to 81.3%, and every other indicator moved in the same direction (Table 6).

The theoretical contribution of the study is the demonstration that ATD and DDR are complementary: ATD supplies the diagnosis (the anatomy of knowledge and of its transposition failures), while DDR supplies the design methodology grounded in that diagnosis. Together they form a single framework that moves from institutional analysis to classroom design.

The practical implications of these findings encompass: (1) the necessity to revise the orientation of mathematics teaching modules to explicitly incorporate the logos block, not merely as an optional explanatory supplement, but as an integral component of every instructional sequence; (2) the significance of teacher professional development that extends beyond mathematical content mastery to encompass the capacity for praxeological analysis and obstacle-based didactical design; and (3) the necessity for assessment reform that incorporates the evaluation of relational justification as an indicator of mathematical competence, alongside procedural correctness.

The limitations discussed above motivate a clear research agenda: replication at larger scale and in other contexts; professional development enabling teachers to implement DDR independently; longitudinal study of the stability of epistemic reorganization; and extension of the framework to other algebraic topics. What is ultimately at stake is the nature of school mathematics itself: whether it consists in mastering procedures that yield correct answers in examinations, or in traversing an epistemic path that yields mathematical knowledge as justified true belief: knowledge that is meaningful, robust, and independently justifiable.

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Declarations

Conflicts of Interest	:	The authors declare no conflict of interest.
Generative AI Statement	:	AI Used for Limited, Non-Substantive Support: Generative AI tools, such as Grammarly and DeepL, were employed solely for language editing and minor phrasing enhancements. All conceptualization, analysis, and scholarly content were independently developed and verified by the authors.
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References

Abou-Hayt, I. (2026). A critical look at the anthropological theory of the didactic. *Research in Mathematics Education, 28*(1), 3–24. https://doi.org/10.1080/14794802.2024.2344754

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J. (2018). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education, 46*(1), 39–87. https://doi.org/10.5951/jresematheduc.46.1.0039

Bosch, M., & Gascón, J. (2006). *25 years of the didactic transposition*. ICMI Bulletin, 58, 51–65.

Bosch, M., & Gascón, J. (2014). Introduction to the Anthropological Theory of the Didactic (ATD). *In A. Bikner-Ahsbahs & S. Prediger (Eds.), Networking of theories as a research practice in mathematics education (pp. 67–84)*. Springer. https://doi.org/10.1007/978-3-319-05389-9_5

Brousseau, G. (1997). *Theory of didactical situations in mathematics (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.)*. Kluwer Academic Publishers.

Bruner, J. S. (1977). *The process of education (2nd ed.)*. Harvard University Press.

Chaachoua, H., Bessot, A., Romo, A., & Castela, C. (2019). Developments and functionalities in the praxeological model. *In M. Bosch, Y. Chevallard, F. J. Garcia, & J. Monaghan (Eds.), Working with the anthropological theory of the didactic: A comprehensive casebook (pp. 41–60)*. Routledge.

Chaachoua, H., Bessot, A., Barquero, B., Pilet, J., Mizoguchi, T., Kaspary, D., & Quoc, N. (2024). A comparative study of the teaching of quadratic equations in five curricula: Brazil, France, Japan, Spain, and Vietnam. *Recherches en Didactique des Mathématiques, 44*(1), 91–135.

Chevallard, Y. (1985). *La transposition didactique: Du savoir savant au savoir enseigné [Didactical transposition: From scholarly knowledge to taught knowledge]*. La Pensée Sauvage.

Chevallard, Y. (2006). Steps towards a new epistemology in mathematics education. In M. Bosch (Ed.), *Proceedings of the IV Congress of the European Society for Research in Mathematics Education (pp. 21–30)*. FUNDEMI-IQS.

Chevallard, Y. (2019). Introducing the Anthropological Theory of the Didactic: An attempt at a principled approach. *Hiroshima Journal of Mathematics Education, 12*, 71–114.

Dubinsky, E. (2001). Using a theory of learning in college mathematics courses. University of Warwick.

Ellis, A. B., & Özgür, Z. (2024). Trends, insights, and developments in research on the teaching and learning of algebra. *ZDM – Mathematics Education, 56*(2), 199–210. https://doi.org/10.1007/s11858-023-01545-9

Erbilgin, E., & Gningue, S. M. (2023). Using the onto-semiotic approach to analyze novice algebra learners' meaning-making processes with different representations. *Educational Studies in Mathematics, 114*(2), 337–357. https://doi.org/10.1007/s10649-023-10247-8

Fardian, D., Suryadi, D., & Prabawanto, S. (2024). Students' learning obstacles in solving mathematical proficiency tasks: A hermeneutic phenomenological study focused on algebra. *Kreano, Jurnal Matematika Kreatif-Inovatif, 15*(2), 593–618. https://doi.org/10.15294/36aygs91

Fardian, D., Suryadi, D., & Prabawanto, S. (2025). A praxeological analysis of linear equations in Indonesian mathematics textbooks: Focusing on systemic and epistemic aspect. *Journal on Mathematics Education, 16*(1), 225–254. https://doi.org/10.22342/jme.v16i1.pp225-254

Jupri, A., & Drijvers, P. (2016). Student difficulties in mathematizing word problems in algebra. *Eurasia Journal of Mathematics, Science and Technology Education, 12*(9), 2481–2502. https://doi.org/10.12973/eurasia.2016.1299a

Kaspary, D., Chaachoua, H., & Bessot, A. (2020). Qu'apporte la notion de portée d'une technique à l'étude de la dynamique praxéologique? *Annales de Didactique et de Sciences Cognitives, 25*, 1–35. https://doi.org/10.4000/adsc.1456

Kieran, C. (2004). Algebraic thinking in the early grades: What is it? *The Mathematics Educator, 8*(1), 139–151.

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *In C. Kieran (Ed.), Early algebra: Research into its nature, its learning, its teaching (pp. 1–22)*. Springer. https://doi.org/10.1007/978-3-030-65683-6

Kieran, C., & Martínez-Hernández, C. (2022). Coordinating invisible and visible sameness within equivalence transformations of numerical equalities by 10- to 12-year-olds in their movement from computational to structural approaches. *ZDM – Mathematics Education, 54*(6), 1215–1227. https://doi.org/10.1007/s11858-022-01355-5

Kullberg, A., Björklund, C., Runesson Kempe, U., Brkovic, I., Nord, M., & Maunula, T. (2024). Improvements in learning addition and subtraction when using a structural approach in first grade. *Educational Studies in Mathematics, 117*(3), 399–417. https://doi.org/10.1007/s10649-024-10339-z

Maudy, S. Y., & Ruli, R. M. (2025). Educators' knowledge transposition: The algebra devolution of thinking. *Jurnal Riset Pendidikan Matematika, 12*(2), 183–201. https://doi.org/10.21831/jrpm.v12i2.84986

Moustakas, C. (1994). *Phenomenological research methods*. Sage.

Pitta-Pantazi, D., Chimoni, M., & Christou, C. (2025). Revisiting the relationship of arithmetical thinking and letter-symbolic algebra. *International Journal of Science and Mathematics Education, 23*(3), 689–711. https://doi.org/10.1007/s10763-024-10493-z

Pittalis, M. (2023). Young students' arithmetic-algebraic structure sense: An empirical model and profiles of students. *International Journal of Science and Mathematics Education, 21*(6), 1865–1887. https://doi.org/10.1007/s10763-022-10333-y

Pritchard, D. (2018). *What is this thing called knowledge? (4th ed.)*. Routledge.

Radford, L. (2021). *The theory of objectification: A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill/Sense.

Radford, L. (2022). Introducing equations in early algebra. *ZDM – Mathematics Education, 54*(6), 1461–1474. https://doi.org/10.1007/s11858-022-01422-x

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1–36. https://doi.org/10.1007/BF00302715

Stephens, A. C., Strachota, S., Knuth, E., Gardiner, A., Blanton, M., & Stroud, R. (2022). What does it mean to 'understand' equations in early algebra? *Educational Studies in Mathematics, 109*(3), 523–545. https://doi.org/10.1007/s10649-021-10090-z

Strømskag, H., & Chevallard, Y. (2026). Didactic transposition and the knowledge to be taught: Towards an archeorganisation for concave/convex functions. *International Journal of Mathematical Education in Science and Technology, 57*(7), 1342–1369. https://doi.org/10.1080/0020739X.2024.2305879

Sumpter, L., Blomqvist, A., Nydahl, A., & Sollerman, S. (2026). Mathematical equivalence and grade 6 students' errors. *Scandinavian Journal of Educational Research, 70*(5), 1112–1123. https://doi.org/10.1080/00313831.2025.2550274

Suryadi, D. (2019). *Landasan filosofis penelitian desain didaktis (DDR) [Philosophical foundations of didactical design research]*. Gapura Press.

Suryadi, D. (2023). *Jalan epistemik menghasilkan pengetahuan melalui Didactical Design Research (DDR) [The epistemic path to knowledge through Didactical Design Research]*. Pusat Pengembangan DDR Indonesia (PUSBANGDDRINDO), Universitas Pendidikan Indonesia.

Veith, J. M., Beste, M. L., Kindervater, M., Krause, M., Straulino, M., Greinert, F., & Bitzenbauer, P. (2023). Mathematics education research on algebra over the last two decades: Quo vadis? *Frontiers in Education, 8*, 1211920. https://doi.org/10.3389/feduc.2023.1211920

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes*. Harvard University Press.

Winsløw, C., Matheron, Y., & Mercier, A. (2014). Study and research courses as an epistemological model for didactics. *Educational Studies in Mathematics, 87*(1), 101–119. https://doi.org/10.1007/s10649-014-9536-2